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Rational Homotopy Theory and Differential Graded Category

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Abstract

We propose a generalization of Sullivan's de Rham homotopy theory to non-simply connected spaces. The formulation is such that the real homotopy type of a manifold should be the closed tensor dg-category of flat bundles on it much the same as the real homotopy type of a simply connected manifold is the de Rham algebra in original Sullivan's theory. We prove the existence of a model category structure on the category of small closed tensor dg-categories and as a most simple case, confirm an equivalence between the homotopy category of spaces whose fundamental groups are finite and whose higher homotopy groups are finite dimensional rational vector spaces and the homotopy category of small closed tensor dg-categories satisfying certain conditions.

Keywords: rational homotopy theory, non-simply connected space, model category, dg-category.

MSC: 55P62, 18G55, 18G30, 18D15, 18D20, 16E45.

1 Introduction

A rationalization of a simply connected space X is a map $f : X \to X_{\mathbb{Q}}$ such that the higher homotopy groups of $X_{\mathbb{Q}}$ are uniquely divisible and f induces an isomorphism $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_n(X_{\mathbb{Q}})$ for each $n \geq 2$. We call the homotopy type of $X_{\mathbb{Q}}$ the rational homotopy type of X and say X is rational if $X_{\mathbb{Q}}$ is homotopy equivalent to X. Sullivan showed rational homotopy type of a simply connected space of finite type can be recovered from its polynominal de Rham algebra and the homotopy category of simply connected rational spaces of finite $\mathbb{Q}$-type are equivalent to the homotopy category of 1-connected commutative dg-$\mathbb{Q}$-algebras of finite type (see [8] or [7] where the authors call the equivalence the Sullivan-de Rham equivalence). A feature of Sullivan's theory is that if one consider a C^∞-manifold, the corresponding dg-algebra over real numbers is (quasi-isomorphic to) the de Rham algebra of the manifold. Because of this feature, Sullivan's theory has geometric applications. See [6, 8, 9].

In the non-simply connected case, as a genaralization of rationalization, Bousfield and Kan [5] constructed a fiberwise rationalization. For a possibly non-simply connected space X, a fiberwise rationalization is a map $f : X \to X_{\mathbb{Q}}$ such that it induces an isomorphism of fundamental groups and the map $\tilde{f} : \widetilde{X} \to \widetilde{X_{\mathbb{Q}}}$ between universal coverings is a rationalization in the above sense. We call the homotopy type of $X_{\mathbb{Q}}$ the rational homotopy type of X. For this notion, A.Gómez-Tato, S.Halperin and D.Tanré [15] generalized the Sullivan's result to non-simply connected spaces. They proposed the notion of local system of commutative cochain algebras as a generalization of commutative dg-algebra and prove rational homotopy type of spaces with finite rank homotopy groups can be recovered from the corresponding local

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system and prove an equivalence theorem for non-simply connected rational spaces with $\mathbb{Q}$-finite dimensional higher homotopy groups. As other generalizations of the Sullivan's theory for non-simply connected spaces, in [11] a more rigid notion of rational homotopy type is studied and in [13] equivariant dg-algebras are used as algebraic models.

In this paper we introduce different algebraic object viewed as a generalization of commutative dg-algebra and as a first step, prove the category of the algebraic objects admits a model category structure (Theorem 1.0.1) and prove an equivalence theorem for rational spaces whose fundamental groups are finite (Theorem 1.0.2). The algebraic objects are small closed tensor differential graded (dg-) categories.

A closed tensor dg-category is, roughly speaking, a dg-category which is equipped with a structure of closed symmetric monoidal category consistent with the differential graded structure (see Definition 2.1.1). If one views a dg-algebra as a dg-category with only one object, a symmetric monoidal structure on a dg-category is a natural generalization of commutativity of dg-algebra and we need to consider closedness of the symmetric monoidal structure. We mainly consider the pointed case and the corresponding augmented objects are closed tensor dg-categories with fiber functors. Here, a fiber functor of a closed tensor dg-category C is a dg-functor from C to $Vect$, the closed tensor category of $\mathbb{Q}$-vector spaces, which preserves closed tensor structures (see Definition 2.3.3).

A feature of our formulation is that if one consider over the real (or complex) numbers, the (closed tensor) dg-category corresponding to a C^∞-manifold is quasi-equivalent to the dg-category of flat bundles on the manifold ([12, section 3]). Here, the dg-category of flat bundles on a manifold M is such that

- its objects are (finite rank) flat vector bundles (V, D) on M, where V is a vector bundle on M and D is a flat connection on V, and
- its complex of morphisms between two flat bundles $(V, D), (V', D')$ is the de Rham complex of M with coefficients in the flat bundle $(\mathfrak{Hom}(V, V'), D_{\mathfrak{Hom}})$, where $\mathfrak{Hom}(V, V')$ is the hom-vector bundle between V and V' and $D_{\mathfrak{Hom}}$ is the flat connection which is induced by D, D'.

This dg-category has natural closed tensor structure. If M is simply connected, all flat bundles are trivial and this dg-category is essentially the same as the de Rham algebra. For rational coefficients, we construct the corresponding dg-category, using polynomial de Rham forms and finite rank rational local systems instead of flat bundles.

1.0.1 Main results

Let $\mathsf{dgCat}_{cl}^{\geq 0}$ be the category of small closed tensor dg-categories. The first main result is the following.

Theorem 1.0.1 (Theorem 2.3.2). *The category $\mathsf{dgCat}_{cl}^{\geq 0}$ has a model category structure where weak equivalences are quasi-equivalences.*

We extract this result from a theorem of Tabuada which states the category of small dg-categories admits a model category structure using Quillen's path-object argument (lifting argument, see Theorem 2.1.3). The main problem is construction of free functor i.e., a left adjoint of the forgetful functor from the category of small closed tensor dg-categories to the category of small dg-categories.

For each pointed simplicial set K, we construct a closed tensor dg-category $T_{dR}(K)$ with a fiber functor. This construction is functorial in the contravariant sense. Let $\mathsf{dgCat}^{\geq 0}_{cl,*}$ be the category of small closed tensor dg-categories with fiber functors. The second main result is the following.

Theorem 1.0.2 (Theorem 3.3.1)**.** *Let $\mathsf{sSet}^{f\mathbb{Q}}_*$ be the category of connected pointed simplicial sets whose fundamental groups are finite and whose higher homotopy groups are uniquely divisible and finite dimensional as $\mathbb{Q}$-vector spaces.*
(1) *There exists a full subcategory Tan^f of $\mathsf{dgCat}^{\geq 0}_{cl,*}$ and the functor $K \mapsto T_{dR}(K)$ induces an equivalence of homotopy categories: $\mathsf{Ho}(\mathsf{sSet}^{f\mathbb{Q}}_*) \simeq \mathsf{Ho}(\mathsf{Tan}^f)^{op}$.*
(2) *Let K be a simplicial set whose fundamental group is finite and whose higher homotopy groups are Abelian groups of finite rank. The adjunction map*

$$K \longrightarrow \mathbb{R}Sp_0 T_{dR}(K),$$

where $\mathbb{R}Sp_0$ is a right adjoint of T_{dR}, is a fiberwise rationalization of K.

The points of the proof of this result are as follows.

- For a finite group G, the unit morphism $K(G,1) \to \mathbb{R}Sp_0 T_{dR}(K(G,1))$ is a weak equivalence of simplicial sets.
- Let L be a simplicial set whose fundamental group is finite and whose higher homotopy groups are finite dimensional rational vector spaces, and $\widetilde{L} \to L \to K(\pi_1(L),1)$ be a homotopy fiber sequence where the map $L \to K(\pi_1(L),1)$ induces an isomorphism of π_1. The corresponding sequence $T_{dR}(K(\pi_1(L),1)) \to T_{dR}(L) \to T_{dR}(\widetilde{L})$ is a homotopy cofiber sequence in the category of closed tensor dg-categories with fiber functors.

In the infinite fundamental group case, we cannot expect the rational homotopy types in the above sense are recovered from the corresponding closed tensor dg-categories. It is likely that these closed tensor dg-categories are equivalent to Toën's schematic homotopy types (see [20, 22] and subsection 1.0.4) but we don't discuss this in the present paper.

1.0.2 Relation with equivariant differential graded algebras

We shall mention the relation between our formulation and the formulation using equivariant dg-algebras (see [13]). An equivariant (commutative) dg-algebra is, by definition, a commutative dg-algebra with a group action. Let us consider the pointed case. Let K be a possibly non-simply connected pointed simplicial set. We take the universal covering $\widetilde{K} \to K$. The corresponding polynomial de Rham algebra $A_{dR}(\widetilde{K})$ has a natural action of $\pi_1(K)$ induced by the action of $\pi_1(K)$ on $\widetilde{K}$. Let $\widetilde{A_{dR}}(K)$ denote the equivariant dg-algebra $(\pi_1(K), A_{dR}(\widetilde{K}))$. Under the finiteness conditions on the higher homotopy groups, one can recover the rational homotopy type of K. In the finite fundamental group case, the closed tensor dg-category $T_{dR}(K)$ and $\widetilde{A_{dR}}(K)$ are equivalent in the following sense. The objects of $T_{dR}(K)$ are by definition, finite rank $\mathbb{Q}$-local systems on K or equivalently, finite dimensional $\mathbb{Q}$-representations of $\pi_1(K)$. Let $\mathbf{1}$ be a trivial 1-dimensional representation and V_r be the regular representation (see definitions below Lemma 3.2.7). Consider the complex of morphisms $A := Hom_{T_{dR}(K)}(\mathbf{1}, V_r)$. The pointwise multiplication $V_r \otimes V_r \to V_r$ induces a structure of commutative dg-algebra on A, the right action of π_1 on V_r induces an action on A and one can see that A is isomorphic to $\widetilde{A_{dR}}(K)$. On the other hand, one can construct a closed tensor dg-category which is equivalent (in the categorical sense) to $T_{dR}(K)$ from $\widetilde{A_{dR}}(K)$.

We have the following diagram of categories.

$$\begin{tikzcd}
 & & (\mathsf{EqdgAlg}^f_{1,*})^{op} \arrow[d, "\mathsf{T}"] \\
\mathsf{sSet}^{f\mathbb{Q}}_* \arrow[urr, "\widetilde{A_{dR}}"] \arrow[rr, "T_{dR}"'] & \arrow[ur, Rightarrow, bend left, "\Phi"] & (\mathsf{Tan}^f)^{op}
\end{tikzcd}$$

Here, $\mathsf{EqdgAlg}^f_{1,*}$ is the category of 1-connected augmented equivariant dg-algebras of finite types (see Definition 3.2.4) and T is a functor. Comparison results are summarized as follows.

Theorem 1.0.3 (Theorem 3.2.10, Proposition 3.2.12)**.**
(1) *There exists a natural transformation*

$$\Phi : T_{dR} \Longrightarrow \mathsf{T} \circ \widetilde{A_{dR}} : \mathsf{sSet}^{f\mathbb{Q}}_* \longrightarrow (\mathsf{dgCat}^{\geq 0}_{cl,*})^{op}$$

such that for each $K \in \mathsf{sSet}^{f\mathbb{Q}}_*$*,* Φ_K *is an equivalence of categories (which underlie closed tensor dg-categories).*
(2) *The functor* T *induces an equivalence of homotopy categories* : $\mathsf{Ho}(\mathsf{EqdgAlg}^f_{1,*}) \xrightarrow{\sim} \mathsf{Ho}(\mathsf{Tan}^f)$

One can prove the functor $\widetilde{A_{dR}}$ induces an equivalence of homotopy categories independently of Theorem 1.0.2 and 1.0.3, though we don't prove in this paper. If we assume this equivalence, Theorem 1.0.2 follows from Theorem 1.0.3. Our way of the proof of Theorem 1.0.2 is not the shortest one but we take the way in order to understand our algebraic objects. In the proof of Theorem 1.0.3, (2) we use internal hom functor.

If the fundamental group is infinite, the closed tensor dg-category is not equivalent to the equivariant dg-algebra in any sense because we consider only finite rank local systems.

1.0.3 Organization of the paper

We review the contents of this paper. The main body is the second and third sections. In the second section, we prove the existence of a model category structure on the category of closed tensor dg-categories (Theorem 1.0.1). In 2.1, we give the definition of closed tensor dg-category and gather known results which is used in the proof. In 2.2 we construct the free functor, which is necessary for the path object argument. In 2.3 we complete the proof.

In the third section we prove the equivalence theorem 1.0.2. In 3.1, we define a Quillen pair between the category of simplicial sets and the opposite category of the category of closed tensor dg-categories. Its left Quillen functor is the above T_{dR}. In 3.2 we compare closed tensor dg-categories with equivariant dg-algebras. We prove Theorem 1.0.3 and more rigid result (see Theorem 3.2.10, (1), (2)). The main tool used here is Tannakian theory summarized in Theorem 3.2.9. We provide some explicit examples of closed tensor dg-categories. We also prove a lemma about homotopy pushout. In 3.3 we prove Theorem1.0.2.

One can read the third section independently of the second section if he or she assume Theorem 2.3.2, Corollary 2.3.4.

Arguments in this paper are all elementary except the language of model category theory.

1.0.4 Background

Our motivation is the application of non-Abelian Hodge theory to the topology of complex projective manifold. In simply connected case, combined with Sullivan's result, Hodge theory gives mixed Hodge structures on rational homotopy groups and rational minimal models of compact Kähler manifolds and complex quasi-projective manifolds and then, the mixed Hodge structures give restrictions to the topology of them.(see [6, 9]). As a generalization of these results to the non-simply connected case, the application of non-Abelian Hodge theory is studied by Katzarkov, Pantev, Toën [21] and Pridham [24, 25]. Non-Abelian Hodge theory states quasi-equivalence between the dg-category of flat bundles and the dg-category of semistable Higgs bundles with vanishing Chern numbers on complex projective manifolds (see [12, Section 3]). In [21, 25] the authors define and construct "mixed Hodge structure" on some algebraic object encoding homotopical data of a complex manifold, by using non-Abelian Hodge theory, then in [21], restrictions to the homotopy types of complex projective manifolds are given and in [25] mixed Hodge structures on real homotopy groups of them are constructed under some assumptions. In [21], the algebraic objects are schematic homotopy types, which are higher stacks and in [25] the ones are the pro-algebraic homotopy types, which are simplicial affine group schemes (see also [23]). Our algebraic objects can be an alternative approach to these problems and we think the use of dg-category is more natural because the dg-category of flat bundles is a natural extension of the de Rham algebra and the dg-category naturally appears in non-Abelian Hodge theory.

1.1 Notation and terminology

Throughout this paper, k denotes a field of characteristic 0 and $\mathbb{Q}$ denotes the field of rational numbers.

1.1.1 dg-categories

All complexes are defined over k and have cohomological grading. $\mathsf{C}(k)$ denotes the symmetric monoidal category of unbounded complexes. By a differential graded (dg-) category, we mean a category enriched over $\mathsf{C}(k)$ (See [19]). $\underline{\mathsf{C}}(k)$ denotes the dg-category of unbounded complexes i.e., the self-enrichment of $\mathsf{C}(k)$. For a (dg-)category C, $Ob(C)$ denotes the set of objects of C and for a (dg-)functor $F : C \to D$, $Ob(F) : Ob(C) \to Ob(D)$ denotes the fuction given by F. If C is a category, $Hom_C(c, c')$ stands for the set of morphisms between c and c'. If C is a dg-category, the same symbol denotes the complex of morphisms. We always identify k-linear categories with dg-categories concentrated in degree 0. Commutative dg-algebra is abbreviated to cdga and dg-category to dgc.

We denote by dgCat the category of small dg-categories and dg-functors between them (see [19]), by $\mathsf{dgCat}^{\geq 0}$ the full subcategory of dgCat consisting of dg-categories C such that $Hom^n_C(c, c') = 0$ for any $c, c' \in Ob(C)$ and for any $n < 0$.

For $C \in \mathsf{dgCat}$ let $Z^0(C)$ (resp. $H^0(C)$) denote the category whose objects are those of C and whose sets of morphisms consist of 0-th cocycles (resp. 0-th cohomology classes) of complexes of morphisms of C. We regard Z^0 and H^0 as functors

$$Z^0, H^0 : \mathsf{dgCat} \longrightarrow \mathsf{Cat},$$

where Cat is the category of small categories. It is clear what Z^0 and H^0 assign to each morphism of dgCat. A morphism in $Z^0(C)$ is said to be a chain morphism in C. A morphism in C is said to be an isomorphism if it is a chain morphism and has an inverse which is also a chain morphism. $Mor(C)$ stands for the set of all homogeneous morphisms of C, i.e., $Mor(C) := \coprod_{(c,c')} \coprod_n Hom^n_C(c, c')$.

Let F, $G : C \to D$ be two dg-functors. A natural transformation $\alpha : F \Rightarrow G$ requires that for each $c \in Ob(C)$ α_c is a chain morphism and compatible with all morphisms of all degrees.

An equivalence (resp. a quasi-equivalence) between dg-categories is a dg-functor which induces an equivalence between Z^0's (resp. H^0's) and isomorphisms (resp. quasi-isomorphisms) of the complexes of morphisms.

Let $C, D \in \mathsf{dgCat}^{\geq 0}$.

- $C \boxtimes D$ denotes a dgc defined as follows.
  - $Ob(C \boxtimes D) = \{(c,d) \mid c \in C,\ d \in D\}$ and
  - $Hom_{C\boxtimes D}((c_0,d_0),(c_1,d_1)) = Hom_C(c_0,c_1) \otimes_k Hom_D(d_0,d_1)$ with the composition given by $(f' \otimes g') \circ (f \otimes g) = (-1)^{\deg g' \cdot \deg f}(f' \circ f) \otimes (g' \circ g)$.
- $\mathcal{T}_{C,D} : C \boxtimes D \to D \boxtimes C$ denotes the morphism defined by $(c,d) \mapsto (d,c)$ and $f \otimes g \mapsto (-1)^{\deg f \cdot \deg g} g \otimes f$.
- C^{op} denotes the opposite dg-category of C whose composition is defined by $g \circ f := (-1)^{\deg g \cdot \deg f} f \circ g$, where the composition of right hand side is that in C.
- We define a dg-functor $Hom_C(-,-) : C^{op} \boxtimes C \to \underline{\mathsf{C}}(k)$ by $Hom_C(c,c') = Hom_C(c,c')$ for $(c,c') \in Ob(C^{op} \boxtimes C)$ and
$$Hom_C(f \otimes g)(\alpha) = (-1)^{\deg f(\deg g + \deg \alpha)} g \circ \alpha \circ f$$
  for $f \otimes g \in Mor(C^{op} \boxtimes C)$.
- $C \times D$ denotes the product in the category $\mathsf{dgCat}^{\geq 0}$. Explicitly, $Ob(C \times D) = Ob(C) \times Ob(D)$, $Hom_{C\times D}((c_0,d_0),(c_1,d_1)) = Hom_C(c_0,c_1) \oplus Hom_D(d_0,d_1)$.

Clearly these constructions are functorial.

We denote by $\mathsf{dgGrph}^{\geq 0}$ the category of non-negatively graded dg-graphs. Its objects are directed graphs whose edges have structures of non-negatively graded complexes and its morphisms are morphisms of directed graphs which induce chain maps on edges. $\mathcal{F}_{cat} : \mathsf{dgGrph}^{\geq 0} \to \mathsf{dgCat}^{\geq 0}$ denotes the free functor of [18, Section 5]. Explicitly, $Ob(\mathcal{F}_{cat}(G)) = Ver(G)$, the set of vertices of G, and
$$Hom_{\mathcal{F}_{cat}(G)}(v,v') := \begin{cases} k \cdot id_v \oplus \bigoplus_{v_1,\ldots,v_l,l\geq 0} Ed(v_l,v') \otimes_k \cdots \otimes_k Ed(v,v_1) & \text{if } v = v' \\ \bigoplus_{v_1,\ldots,v_l \geq 0} Ed(v_l,v') \otimes_k \cdots \otimes_k Ed(v,v_1) & \text{otherwise} \end{cases}$$
where Ed stands for the complex of edges. $\mathcal{F}_{cat}$ is a left adjoint of the forgetful functor $\mathsf{dgCat}^{\geq 0} \to \mathsf{dgGrph}^{\geq 0}$.

A non-unital dg-category is a dg-graph G which is equipped with associative composition $Ed(v',v'') \otimes_k Ed(v,v') \to Ed(v,v'')$ for each $v,v',v'' \in Ver(G)$ (but without units). An ideal of a dg-category C is a non-unital dg-subcategory I of C such that it contains all objects of C and if f and g are morphisms of C, one of them is in I, and $g \circ f$ exists, $g \circ f$ is in I. If I is an ideal of C, then a dg-category C/I is defined by $Ob(C/I) = Ob(C)$ and $Hom_{C/I}(c,c') = Hom_C(c,c')/Hom_I(c,c')$. For a subset S of $Mor(C)$, the ideal generated by S is the smallest ideal which contains S.

1.1.2 Model categories

Our notion of model category is that of [14]. Let M be a model category.

Let $\emptyset \in \mathsf{M}$ be an initial objects. The over category $\mathsf{M}/\emptyset$ has a model category structure where equivalences, fibrations and cofibrations are detected by underlying morphisms of M (see [14, Remark 1.1.7, Proposition 1.1.8]). We always regard the category $\mathsf{M}/\emptyset$ as a model category by this structure.

The notions of path object and right homotopy are found in [14, DEFINITION 1.2.4].

The notion of homotopy pushout squares in M is found in [14, p.184] and if M is a pointed category, a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms of M with $g \circ f = 0$ is said to be a homotopy cofiber sequence if the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

is a homotopy pushout square. Here $*$ is a terminal (and initial) object of M. The notion of homotopy fiber sequences is its dual notion.

$\mathsf{Ho}(\mathsf{M})$ denotes the homotopy category of M. Let M' be a full subcategory of M stable under weak equivalences of M. We denote by $\mathsf{Ho}(\mathsf{M}')$ the full subcategory of $\mathsf{Ho}(\mathsf{M})$ consisting of objects of M'. It is easy to see $\mathsf{Ho}(\mathsf{M}')$ is (isomorphic to) the localization of M' obtained by inverting weak equivalences in M'

sSet stands for the category of simplicial sets. We regard sSet as a model category with the usual model structure (see [14]). sSet_* stands for the category of pointed simplicial sets. sSet_*^f (resp. $\mathsf{sSet}_*^{f\mathbb{Q}}$) denotes the full subcategory of sSet_* consisting of connected K with $\pi_1(K)$ finite, $\pi_n(K)$ being an Abelian group of finite rank (resp. with $\pi_1(K)$ finite, $\pi_n(K)$ uniquely divisible and finite dimensional as a $\mathbb{Q}$-vector space) for each $n \geq 2$. Let $K \in \mathsf{sSet}$. $\Delta(K)$ denotes the category of simplices of K of [14, Chapter 3]:

- An object of $\Delta(K)$ is a simplex of K, i.e., $Ob(\Delta(K)) = \coprod_{n \geq 0} K_n$,
- for $\sigma \in K_n$, $\tau \in K_m$ a morphism $a : \sigma \to \tau$ is a morphism $a : [n] \to [m] \in \Delta$ such that $a^*(\tau) = \sigma$

where Δ is the category with objects $[l] = \{0, \dots, l\}$ for $l \geq 0$, and weakly order-preserving maps.

2 Model of homotopy theory of closed tensor dg-categories

The purpose of this section is to give a model structure on the category of small closed tensor dg-categories (see Theorem 2.3.2). This result is a foundation of arguments of next section.

2.1 Preliminaries

2.1.1 Closed tensor dg-categories

The following definition is a variant of the usual definition of closed symmetric monoidal category (see [14, Definition 4.1.12] or [3, P.180], where the author call it closed category) in the differential graded context.

Definition 2.1.1. (1) *Let C be an object of* $\mathsf{dgCat}^{\geq 0}$. *A* closed tensor structure *on C is a 11-tuple*

$$((-\otimes-),\ \mathbf{1},\ a,\ \tau,\ u,\ \mathfrak{Hom},\ \phi,\ (-\oplus-),\ s_1,\ s_2,\ \mathbf{0})$$

consisting of

- *a morphism* $(-\otimes-) : C \boxtimes C \longrightarrow C \in \mathsf{dgCat}^{\geq 0}$,
- *a distinguished object* $\mathbf{1} \in C$,
- *natural isomorphisms*

$$\begin{aligned} a : ((-\otimes-)\otimes-) &\Longrightarrow (-\otimes(-\otimes-)) &&: (C\boxtimes C)\boxtimes C \cong C\boxtimes(C\boxtimes C) \longrightarrow C, \\ \tau : (-\otimes-) &\Longrightarrow (-\otimes-)\circ \mathcal{T}_{C,C} &&: C\boxtimes C \longrightarrow C, \\ u : (-\otimes\mathbf{1}) &\Longrightarrow id_C &&: C \longrightarrow C \end{aligned}$$

  satisfying usual coherence conditions on associativity, commutativity and unity, see [3, pp.251],

- *a morphism* $\mathfrak{Hom} : C^{op}\boxtimes C \longrightarrow C \in \mathsf{dgCat}^{\geq 0}$,
- *a natural isomorphism*

$$\phi : Hom_C(-\otimes-,-) \Longrightarrow Hom_C(-,\mathfrak{Hom}(-,-)) : C^{op}\boxtimes C^{op}\boxtimes C \longrightarrow \mathsf{C}(k),$$

- *a morphism* $(-\oplus-) : C\times C \longrightarrow C \in \mathsf{dgCat}^{\geq 0}$,
- *two natural transformations*

$$P_1 \overset{s_1}{\Longrightarrow} (-\oplus-) \overset{s_2}{\Longleftarrow} P_2 : C\times C \longrightarrow C,$$

  where $P_i : C\times C \longrightarrow C$ is the i-th projection, such that the induced morphism $s_1^*\times s_2^* : Hom_C(c_0\oplus c_1, c') \longrightarrow Hom_C(c_0,c')\times Hom_C(c_1,c')$ is an isomorphism (i.e., $c_0\oplus c_1$ is a coproduct), and

- *a distinguished object $\mathbf{0}\in Ob(C)$ such that $Hom_C(\mathbf{0},c) = 0$ for any $c\in Ob(C)$.*

We call $(-\otimes-)$ *a* tensor functor *and* $\mathfrak{Hom}$ *a* internal hom functor.
(2) *A* closed tensor dg-category *is an object C of* $\mathsf{dgCat}^{\geq 0}$ *equipped with a closed tensor structure. For two closed tensor dg-categories C, D, a* morphism of closed tensor dg-categories *is a morphism $F : C\to D$ of dgc's which preserves all of the above structures. For example, $F(c\otimes d) = F(c)\otimes F(d)$ (not only naturally isomorphic), $F(\tau_{c,c'}) = \tau_{Fc,Fc'}$ and $F(\mathbf{1}) = \mathbf{1}$. We denote by* $\mathsf{dgCat}_{cl}^{\geq 0}$ *the category of closed tensor dgc's. A* closed tensor category *is a closed tensor dg-category whose complexes of morphisms are concentrated in degree 0 and a* morphism of closed tensor categories *is the same as a morphism of closed tensor dg-categories.*

We apply the notions of equivalence and quasi-equivalence to objects of $\mathsf{dgCat}_{cl}^{\geq 0}$ via the forgetful functor $\mathsf{dgCat}_{cl}^{\geq 0} \to \mathsf{dgCat}^{\geq 0}$. Note that an equivalence in $\mathsf{dgCat}_{cl}^{\geq 0}$ doesn't always have a quasi-inverse which is a morphism of $\mathsf{dgCat}_{cl}^{\geq 0}$. We say two objects of $\mathsf{dgCat}_{cl}^{\geq 0}$ are equivalent if they can be connected by a finite chain of equivalences in $\mathsf{dgCat}_{cl}^{\geq 0}$.

Let $T : C\boxtimes C$ (resp. $C^{op}\boxtimes C$) $\to C$ be a dg-functor. A T-closed ideal is an ideal I of C such that $T(f,g)$ is in I if one of f and g is in I. If I is T-closed ideal, T induces a functor $\overline{T} : (C/I)\boxtimes(C/I)$ (resp. $(C/I)^{op}\boxtimes(C/I)) \to C/I$. There is an obvious notion of T-closed ideal generated by S.

2.1.2 A model category structure on dgCat

The following theorem is the main result of [16] which is crucial for our argument.

Theorem 2.1.2 ([16])**.** *The category* dgCat *has a cofibrantly generated model structure where weak equivalences and fibrations are defined as follows.*

- *A morphism* $F : C \to D \in \mathsf{dgCat}$ *is a weak equivalence if and only if it is a quasi-equivalence.*
- *A morphism* $F : C \to D \in \mathsf{dgCat}$ *is a fibration if and only if it satisfies the following two conditions.*
  - *For* $c, c' \in Ob(C)$ *the morphism* $F_{(c,c')} : Hom_C(c, c') \to Hom_D(Fc, Fc')$ *is an epimorphism.*
  - *For any* $c \in Ob(C)$ *and any isomorphism* $f : Fc \to d' \in H^0(D)$*, there exists an isomorphism* $g : c \to c' \in H^0(C)$ *such that* $H^0(F)(g) = f$*.*

2.1.3 Path object argument

The path object argument is due to Quillen [2] and the following form is found in [17, section 5].

Theorem 2.1.3 (Path object argument, [2, 17])**.** *Let* M *be a category with all small limits and colimits and* N *be a cofibrantly generated model category. Let* $\mathcal{U} : \mathsf{M} \longrightarrow \mathsf{N}$ *be a functor which commutes with all filtered colimits. Suppose that*

- *all objects of* N *are fibrant,*
- $\mathcal{U}$ *possesses a left adjoint* $\mathcal{L} : \mathsf{N} \longrightarrow \mathsf{M}$*, and*
- *There exists an endo-functor* $\mathcal{P} : \mathsf{M} \longrightarrow \mathsf{M}$ *and a sequence of natural transformations*

$$Id_{\mathsf{M}} \xRightarrow{i} \mathcal{P} \xRightarrow{d_0 \times d_1} \Pi : \mathsf{M} \longrightarrow \mathsf{M},$$

  where Π *is defined by* $\Pi(X) = X \times X$*, such that for each object* $X \in \mathsf{M}$ *the composition* $(d_0 \times d_1)_X \circ i_X$ *is the diagonal* $X \to X \times X$ *and the diagram* $\mathcal{U}X \overset{\mathcal{U}i}{\to} \mathcal{U}\mathcal{P}X \overset{\mathcal{U}d_0 \times \mathcal{U}d_1}{\to} \mathcal{U}\Pi X = \mathcal{U}X \times \mathcal{U}X$ *is a path object in the sense of [14, Definition 1.2.4].*

Then, there exists a cofibrantly generated model category structure on M *such that a morphism* $f : X \to Y \in \mathsf{M}$ *is a weak equivalence (resp. a fibration) if and only if* $\mathcal{U}f : \mathcal{U}X \to \mathcal{U}Y$ *is a weak equivalence (resp. a fibration) in* N*.*

We call a functor $\mathcal{P}$ satisfying the above condition a functorial path object.

2.2 Free construction

In order to use path object argument, we shall construct a left adjoint

$$\mathcal{F}_{cl}^{\geq 0} : \mathsf{dgCat} \longrightarrow \mathsf{dgCat}_{cl}^{\geq 0}.$$

of the forgetful functor $\mathcal{U} : \mathsf{dgCat}_{cl}^{\geq 0} \to \mathsf{dgCat}$. We call this functor the free functor. (We use the definite article in the sense that it is unique up to natural isomorphisms.) We divide construction of $\mathcal{F}_{cl}^{\geq 0}$ into construction of two functors,

$$\mathcal{T}^{\geq 0} : \mathsf{dgCat} \longrightarrow \mathsf{dgCat}^{\geq 0} \quad \text{and} \quad \mathcal{F}_{cl} : \mathsf{dgCat}^{\geq 0} \longrightarrow \mathsf{dgCat}_{cl}^{\geq 0}.$$

Here, $\mathcal{T}^{\geq 0}$ is a left adjoint of the inclusion functor $\mathcal{I} : \mathsf{dgCat}^{\geq 0} \to \mathsf{dgCat}$ and $\mathcal{F}_{cl}$ is a left adjoint of the forgetful functor: $\mathsf{dgCat}^{\geq 0}_{cl} \to \mathsf{dgCat}^{\geq 0}$, which we call the free functor too. If these two functors exist, it is clear that $\mathcal{F}^{\geq 0}_{cl} = \mathcal{F}_{cl} \circ \mathcal{T}^{\geq 0}$.

We first define $\mathcal{T}^{\geq 0}$. Let $C \in \mathsf{dgCat}$. For each $c, c' \in Ob(C)$, let $M(c,c') \subset Hom_C(c,c')$ be the subcomplex generated by homogeneous elements of the form

$$\sum_i f_{i,1} \circ \cdots \circ f_{i,k_i},$$

where each $f_{i,j}$ is a homogeneous morphism and for each i, at least one of $f_{i,j}$'s has negative degree. We put

$$Ob(\mathcal{T}^{\geq 0}C) = Ob(C), \qquad Hom_{\mathcal{T}^{\geq 0}C}(c,c') = Hom_C(c,c')/M(c,c')$$

and define the composition of $\mathcal{T}^{\geq 0}C$ from that of C. One can easily see the construction $C \longmapsto \mathcal{T}^{\geq 0}C$ is functorial and satisfies the required property.

The idea to construct $\mathcal{F}_{cl} : \mathsf{dgCat}^{\geq 0} \to \mathsf{dgCat}^{\geq 0}_{cl}$ is also elementary: Attaching necessary objects and morphisms and taking quotients by necessary relations. To make this precise, we first define the following:

2.2.1 Universal dg-categories

In this sub-subsection, we define a dgc which is initial among the dgc's having given objects, given morphisms, given relations, and a morphism from given dgc. Suppose the following data are given:

- a set S_{ob},
- a set S_{mor} of non-negatively graded complexes (i.e., S_{mor} is a subset of $Ob(\mathsf{C}(k))$).
- two functions s, $t : S_{mor} \to S_{ob}$ which we call the source function and target function, respectively,
- a dgc C and a function $\mathfrak{o} : Ob(C) \to S_{ob}$.

Let $\bigsqcup S_{mor}$ be the set of homogeneous elements of complexes belonging to S_{mor}, i.e., $\bigsqcup S_{mor} = \bigsqcup_{n\geq 0} \bigsqcup_{H \in S_{mor}} H^n$.

Let $Mor(C, S_{mor})$ be the set of "formal morphisms generated by $Mor(C)$ and $\bigsqcup S_{mor}$". More precisely, an element of the set $Mor(C, S_{mor})$ is a formal linear combination of formal compositions:

$$\sum_n c_n(\alpha_{n,k_n} \circ \cdots \circ \alpha_{n,1})$$

such that $c_n \in k$ and each $\alpha_{n,i}$ is an element of the union

$$Mor(C) \sqcup \bigsqcup S_{mor} \sqcup \{id_x\}_{x \in S_{ob}}$$

(id_x is a formal symbol), $t(\alpha_{n,i}) = s(\alpha_{n,i+1})$ for each n and each $i = 1, \dots k_n - 1$, and $t(\alpha_{n,k_n}) = t(\alpha_{n',k_{n'}})$, $s(\alpha_{n,1}) = s(\alpha_{n',1})$ for each n, n'. Here, if α is a morphism in C, $s(\alpha)$ (resp. $t(\alpha)$) denotes the image of the source of α (resp. the target of α) by $\mathfrak{o}$.

Suppose the following additional datum is given:

- S_{rel}, a subset of $Mor(C, S_{mor})$

Definition 2.2.1. *With the above notations, the* universal dgc associated to $(C, S_{ob}, S_{mor}, S_{rel}, s, t, \mathfrak{o})$ *is a 5-tuple*

$$(D,\ I_D,\ \mathfrak{o}_D,\ \mathfrak{m}_D,\ \{f_{D,H}\}_{H\in S_{mor}})$$

consisting of

- *a dgc D, written $C[S_{ob}, S_{mor}]/S_{rel}$,*
- *a morphism of dgc's $I_D : C \to D$,*
- *a function $\mathfrak{o}_D : S_{ob} \to Ob(D)$,*
- *a function $\mathfrak{m}_D : S_{mor} \to \underline{Mor}(D)$ where $\underline{Mor}(D) = \{\, Hom_D(x, y) \,|\, (x, y) \in Ob(D)^{\times 2} \,\}$, and*
- *a family of homomorphisms of complexes $\{f_{D,H} : H \to \mathfrak{m}_D(H)\}_{H\in S_{mor}}$*

satisfying the following conditions.

- *It must satisfy obvious consistency conditions. Firstly, the diagram*

$$\begin{array}{ccc} S_{mor} & \xrightarrow{\mathfrak{m}_D} & \underline{Mor}(D) \\ {\scriptstyle s\ (resp.\ t)}\big\downarrow & & \big\downarrow{\scriptstyle s_D\ (resp.\ t_D)} \\ S_{ob} & \xrightarrow{\mathfrak{o}_D} & Ob(D), \end{array}$$

  where the right vertical arrow is the source function (resp. the target function) of D, commutes.
- *Secondly, the composition $Ob(C) \xrightarrow{\mathfrak{o}} S_{ob} \xrightarrow{\mathfrak{o}_D} Ob(D)$ is equal to the function $Ob(I_D) : Ob(C) \to Ob(D)$.*
- *The function $Mor(C, S_{mor}) \to Mor(D)$ defined from I_D, $\mathfrak{m}_D$ and $f_{D,H}$'s takes S_{rel} to zeros.*
- *It has a universal property. If a 5-tuple*

$$(E,\ I_E : C \to E,\ \mathfrak{o}_E : S_{ob} \to Ob(E),\ \mathfrak{m}_E : S_{mor} \to \underline{Mor}(E),\ \{f_{E,H} : H \to \mathfrak{m}_E(H)\})$$

  which satisfies all of the above conditions where D is replaced with E is given, there exists a unique morphism of dgc's $D \to E$ which preserves all of the above structures.

We shall construct the universal dgc. If we have constructed such a dgc for the case $S_{rel} = \emptyset$, then for general case, we only have to put $C[S_{ob}, S_{mor}]/S_{rel} = D_0/I$ where $D_0 = C[S_{ob}, S_{mor}]/\emptyset$ and I is the ideal of D_0 generated by the image of S_{rel} by the function $Mor(C, S_{mor}) \to Mor(D_0)$. So we may assume $S_{rel} = \emptyset$.

We first define a dg-graph A by

- $Ver(A) = S_{ob}$,
- $Ed(x, y) = \displaystyle\bigoplus_{\substack{H\in S_{mor},\\ s(H)=x,\ t(H)=y}} H \ \oplus \bigoplus_{\mathfrak{o}(c)=x, \mathfrak{o}(c')=y} Hom_C(c, c')$.

We consider the dgc $D' := \mathcal{F}_{cat}(A)$. Let $[f]$ denote the morphism of $\mathcal{F}_{cat}(A)$ corresponding to an edge $f \in A$. Let J be the ideal of D' generated by

$$R = \{\, [id_c] - id_{\mathfrak{o}(c)},\ [g \circ f] - [g] \circ [f] \,|\, c \in Ob(C),\ f,\ g \in Mor(C) \,\}.$$

Put $D = D'/J$. The set of relations R ensures that one can define a dg-functor $C \to D$ by $c \mapsto \mathfrak{o}(c)$ and $f \mapsto [f]$. The other data are defined obviously and it is clear that D is the required universal dgc.

2.2.2 Free closed tensor dg-categories

We shall construct the free functor

$$\mathcal{F}_{cl} : \mathsf{dgCat}^{\geq 0} \longrightarrow \mathsf{dgCat}^{\geq 0}_{cl},$$

i.e., a left adjoint of the forgetful functor.

Let S be a set and $W_{cl}(S)$ be the set of words generated by S and formal symbols $\mathbf{1}$, $\mathbf{0}$ with operations $\otimes$, $\mathfrak{Hom}$ and $\oplus$. More precisely, $W_{cl}(S)$ is defined inductively as follows. Set

$$\begin{aligned}
W^0_{cl}(S) &= S \sqcup \{\mathbf{1}, \mathbf{0}\},\\
W^p_{cl}(S) &= \{\, x \otimes y,\ \mathfrak{Hom}(x, y),\ x \oplus y \mid x,\ y \in W^{p-1}_{cl}(S) \,\} \sqcup W^0_{cl}(S).
\end{aligned}$$

Let $i_0 : W^0_{cl}(S) \to W^1_{cl}(S)$ be the inclusion and $i_p : W^p_{cl}(S) \to W^{p+1}_{cl}(S)$ be the map defined inductively by $i_p(x \otimes y) = i_{p-1}x \otimes i_{p-1}y$, $i_p(\mathfrak{Hom}(x, y)) = \mathfrak{Hom}(i_{p-1}x, i_{p-1}y)$ and $i_p(x \oplus y) = i_{p-1}x \oplus i_{p-1}y$. We identify $W^p_{cl}(S)$ with a subset of $W^{p+1}_{cl}(S)$ via i_p and set

$$W_{cl}(S) := \bigcup_{p \geq 0} W^p_{cl}(S).$$

Note that for example, the operation $\otimes$ doesn't satisfy the associativity low and $\mathbf{1}$ doesn't play any special role yet.

Let $C \in \mathsf{dgCat}^{\geq 0}$. To construct the free closed tensor dg-category $\mathcal{F}_{cl}(C)$ associated to C, we first construct the following **data**:

- a sequence of dgc's $C = D_{-1} \xrightarrow{I_{-1}} \cdots \xrightarrow{I_{j-1}} D_j \xrightarrow{I_j} \cdots$ such that $Ob(D_j) = W_{cl}(Ob(C))$ for $j \geq 0$,
- a morphism of dgc's $T_j : D_j \boxtimes D_j \to D_{j+1}$ for each $j \geq -1$,
- isomorphisms $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, $\tau_{x,y} : x \otimes y \to y \otimes x \in Mor(D_0)$ whose inverse is $\tau_{y,x}$, and $u_x : x \otimes \mathbf{1} \to x \in Mor(D_0)$ for each x, y and $z \in Ob(D_0)$
- a morphism of dgc's $\mathfrak{H}_j : D^{op}_j \boxtimes D_j \to D_{j+1}$ for each $j \geq -1$ and
- chain morphisms $ev^y_x : \mathfrak{Hom}(x, y) \otimes x \to y$ and $coev^y_x : x \to \mathfrak{Hom}(y, x \otimes y)$ in D_0, which we call an evaluation and a coevaluation, for each x, $y \in D_0$,
- chain morphisms $s^{x_0,x_1}_i : x_i \to x_0 \oplus x_1$, $p^{x_0,x_1}_i : x_0 \oplus x_1 \to x_i$ for $x_i \in Ob(D_0)$, for $i = 0, 1$

which satisfy the following **conditions**:

- $Ob(I_{-1})$ is the inclusion $Ob(C) \to W_{cl}(Ob(C))$ and for $j \geq 0$ $Ob(I_j)$ is the identity.
- For $j \geq -1$, $Ob(T_j) = \otimes$ and $Ob(\mathfrak{H}_j) = \mathfrak{Hom}$ where the right hand side of each equation is the operation of $W_{cl}(Ob(C))$.
- For $j \geq -1$, $I_{j+1} \circ T_j = T_{j+1} \circ (I_j \boxtimes I_j) : D_j \boxtimes D_j \to D_{j+2}$.
- For $j \geq -1$, $I_{j+1} \circ \mathfrak{H}_j = \mathfrak{H}_{j+1} \circ (I^{op}_j \boxtimes I_j) : D^{op}_j \boxtimes D_j \to D_j$.

- For $j \geq -1$, $(I_j, T_j, \mathfrak{H}_j)$ has a *universal property* as follows. Suppose the following data are given.
  - a closed tensor dgc E,
  - a morphism of dgc's $I' : D_j \to E$.

  Suppose also these data satisfy the following conditions.
  - $I' \circ T_{j-1} = (- \otimes_E -) \circ (I' \boxtimes I') \circ (I_{j-1} \boxtimes I_{j-1}) : D_{j-1} \boxtimes D_{j-1} \longrightarrow E.$
  - $I' \circ \mathfrak{H}_{j-1} = \mathfrak{Hom}_E \circ (I'^{op} \boxtimes I') \circ (I^{op}_{j-1} \boxtimes I_{j-1}) : (D_{j-1})^{op} \boxtimes D_{j-1} \longrightarrow E.$
  - I' takes a's, τ's, u's, ev's, $coev$'s, s_i's, p_i's $\mathbf{1}$, and $\mathbf{0}$ to the corresponding morphisms and objects.

  (If $j = -1$, these conditions are ignored.) Then if $j \geq 0$ (resp. $j = -1$), there is a unique morphism $\tilde{I}' : D_{j+1} \to E$ such that $\tilde{I}' \circ I_j = I'$, $\tilde{I}' \circ T_j = (- \otimes_E -) \circ (I' \boxtimes I')$ and $\tilde{I}' \circ \mathfrak{H}_j = \mathfrak{Hom}_E \circ (I'^{op} \boxtimes I')$ (resp. $\tilde{I}'$ takes a's, τ's, u's, ev's, $coev$'s, s_i's, p_i's $\mathbf{1}$, and $\mathbf{0}$ to the corresponding morphisms and objects).

Construction: The construction of the above data proceeds in induction. Suppose we have constructed the stage $p \geq -1$. In other words, we have constructed

- a sequence of dgc's $D_{-1} \overset{I_{-1}}{\to} \cdots \overset{I_{p-1}}{\to} D_p$,
- a morphism of dgc's $T_j : D_j \boxtimes D_j \to D_{j+1}$ for each $j \leq p - 1$ and
- a morphism of dgc's $\mathfrak{H}_j : D_j^{op} \boxtimes D_j \to D_{j+1}$ for each $j \leq p - 1$

which satisfy the above conditions. We shall construct D_{p+1}. The idea is to define S_{ob}, S_{mor} and S_{rel} appropriately, and put $D_{p+1} = D_p[S_{ob}, S_{mor}]/S_{rel}$.

Set $S_{ob} := W_{cl}(Ob(C))$ and $\mathfrak{o} := id_{W_{cl}(Ob(C))}$. We shall define S_{mor}.
Notation: In the following, for a set of complexes M (which will be a subset of S_{mor}) we use the expression

$$M = \{\, M(x, y) \mid (x, y) \in M_{ob} \,\}.$$

This means M consists of $M(x, y)$'s, $M(x, y)$ is a complex whose source (resp. target) is x (resp. y) and (x, y) runs through M_{ob}, which is a subset of $(S_{ob})^{\times 2}$.

When $p = -1$, We define sets of complexes

$$Tens,\ Ass,\ Ass^{-1},\ Comm, Unit,\ Unit^{-1},\ Int,\ Ev,\ Coev,\ Inc_i,\ Proj_i\ \ (i = 1, 2).$$

Put

$$\begin{aligned} Tens(x_1 \otimes x_2, y_1 \otimes y_2) &= Hom_{D_p}(x_1, y_1) \otimes_k Hom_{D_p}(x_1, y_1), \\ Int(\,\mathfrak{Hom}(x_1, x_2),\, \mathfrak{Hom}(y_1, y_2)\,) &= Hom_{D_p}(y_1, x_1) \otimes_k Hom_{D_p}(x_2, y_2) \end{aligned}$$

$$\begin{aligned} Ass(\,(x_1 \otimes x_2) \otimes x_3,\, x_1 \otimes (x_2 \otimes x_3)\,) &= k \cdot a_{x_1, x_2, x_3}, \\ Comm(x_1 \otimes x_2, x_2 \otimes x_1) &= k \cdot \tau_{x_1, x_2}, \\ Unit(x \otimes \mathbf{1}, x) &= k \cdot u_x, \\ Ev(\mathfrak{Hom}(x, y) \otimes x, y) &= k \cdot ev_x^y, \\ Coev(x, \mathfrak{Hom}(y, x \otimes y)) &= k \cdot coev_x^y, \\ Inc_i(x_i, x_0 \oplus x_1) &= k \cdot s_i^{x_0, x_1}, \\ Proj_i(x_0 \oplus x_1, x_i) &= k \cdot p_i^{x_0, x_1}. \end{aligned}$$

It will be clear that what $Tens_{ob}, \ldots, Coev_{ob}$ are. For example,

$$Tens_{ob} = \{\, (x_1 \otimes x_2, y_1 \otimes y_2) \mid x_1,\, x_2,\, y_1,\, y_2 \in S_{ob}\, \}.$$

We agree that Ass^{-1} (resp. $Unit^{-1}$) is a copy of Ass (resp. $Unit$) whose source and target functions are replaced with each other and we denote by a'_{x_1, x_2, x_3} (resp. u'_x) the element of the complex belonging to Ass^{-1} (resp. $Unit^{-1}$) which corresponds to a_{x_1, x_2, x_3} (resp. u_x). We set

$$\begin{aligned} S_{mor} =& Tens \sqcup Ass \sqcup Ass^{-1} \sqcup Comm \sqcup Unit \sqcup Unit^{-1} \sqcup Int \sqcup Ev \sqcup Coev \\ & \sqcup Inc_1 \sqcup Inc_2 \sqcup Proj_1 \sqcup Proj_2. \end{aligned}$$

To define S_{rel}, we define three subsets of $Mor(D_p, S_{mor})$, R_1, R_2 and R_3 as follows. In the following, $T'_p(f_1, f_2)$ (resp. $\mathfrak{H}'_p(f_1, f_2)$) denotes the element $f_1 \otimes f_2 \in Tens(x_1 \otimes x_2, y_1 \otimes y_2)$ (resp. $\in Int(\,\mathfrak{Hom}(x_1, x_2),\, \mathfrak{Hom}(y_1, y_2)\,)$).

$$R_1 = \left\{ \begin{array}{c} T'_p(g_1, g_2) \circ T'_p(f_1, f_2) - (-1)^{\deg g_2 \cdot \deg f_1} T'_p(g_1 \circ f_1, g_2 \circ f_2), \\ id_{x_1 \otimes x_2} - T'_p(id_{x_1}, id_{x_2}) \end{array} \right\},$$

$$R_2 = \left\{ \begin{array}{c} \mathfrak{H}'_p(g_1, g_2) \circ \mathfrak{H}'_p(f_1, f_2) - (-1)^{(\deg g_1 + \deg g_2) \deg f_1} \mathfrak{H}'_p(f_1 \circ g_1, g_2 \circ f_2), \\ id_{\mathfrak{Hom}(x_1, x_2)} - \mathfrak{H}'_p(id_{x_1}, id_{x_2}) \end{array} \right\},$$

$$R_3 = \left\{ \begin{array}{c} a'_{x_1, x_2, x_3} \circ a_{x_1, x_2, x_3} - id_{(x_1 \otimes x_2) \otimes x_3},\ a_{x_1, x_2, x_3} \circ a'_{x_1, x_2, x_3} - id_{x_1 \otimes (x_2 \otimes x_3)}, \\ u'_x \circ u_x - id_{x \otimes \mathbf{1}},\ u_x \circ u'_x - id_x,\ \tau_{x_2, x_1} \circ \tau_{x_1, x_2} - id_{x_1 \otimes x_2} \end{array} \right\}.$$

We set $S_{rel} = R_1 \sqcup R_2 \sqcup R_3$ and $D_{p+1} := D_p[S_{ob}, S_{mor}]/S_{rel}$. $I_p : D_p \to D_{p+1}$ is the structure morphism of the universal dgc. R_1 (resp. R_2) ensures that one can define a dg-functor $T_p : D_p \boxtimes D_p \to D_{p+1}$ (resp. $\mathfrak{H}_p : D_p^{op} \boxtimes D_p \to D_{p+1}$) by $f_1 \otimes f_2 \mapsto T'_p(f_1, f_2)$ (resp. $f_1^{op} \otimes f_2 \mapsto \mathfrak{H}'_p(f_1, f_2)$). R_3 ensures that a_{x_1, x_2, x_3},

τ_{x_1,x_2} and u_x are isomorphisms.

When $p \geq 0$, We put $S_{mor} = Tens \sqcup Int$ where $Tens$ and Int are defined by the same formula as above. To define S_{rel} we define four sets of relations R_1, R_2, R'_3, and R'_4. R_1 and R_2 are the ones defined above and

$$\begin{aligned} R'_3 &= \{\, T'_p(I_{p-1}(f_1), I_{p-1}(f_2)) - T_{p-1}(f_1 \otimes f_2)\,\}, \\ R'_4 &= \{\, \mathfrak{H}'_p(I_{p-1}(f_1), I_{p-1}(f_2)) - \mathfrak{H}_{p-1}(f_1^{op} \otimes f_2)\,\}. \end{aligned}$$

We set $S_{rel} = R_1 \sqcup R_2 \sqcup R'_3 \sqcup R'_4$ and $D_{p+1} := D_p[S_{ob}, S_{mor}]/S_{rel}$. R'_3 and R'_4 ensure the compatibility involving I_-, T_- and $\mathfrak{H}_-$ and we have completed the induction. □

Now, we shall define the free closed tensor dgc $\mathcal{F}_{cl}(C)$ associated to C. Set

$$\begin{aligned} D' &= \mathrm{colim}_j\,(I_j, D_j), \\ T' &= \mathrm{colim}_j\,T_j : D' \boxtimes D' \to D', \\ \mathfrak{H}' &= \mathrm{colim}_j\,\mathfrak{H}_j : D'^{op} \boxtimes D' \to D', \end{aligned}$$

where we regard $D' \boxtimes D' \cong \mathrm{colim}_j(D_j \boxtimes D_j)$ and $D'^{op} \boxtimes D' \cong \mathrm{colim}_j(D_j^{op} \boxtimes D_j)$. These are well-defined by the compatibility of I_j's with T_j's and of I_j's with $\mathfrak{H}_j$'s. Let J be the T'-closed and $\mathfrak{H}'$-closed ideal generated by the relations which ensure the following conditions.

- a, τ and u are natural isomorphisms and all of the coherence diagrams required in the definition of closed tensor dgc are commutative.
- The morphisms of complex $\phi_{x,y,z} : Hom_{D'}(x \otimes y, z) \to Hom_{D'}(x, \mathfrak{Hom}(y,z))$ given by $f \mapsto \mathfrak{H}'(id_y \otimes f) \circ coev_x^y$ form a natural isomorphism whose inverse $\varphi_{x,y,z} : Hom_{D'}(x, \mathfrak{Hom}(y,z)) \to Hom_{D'}(x \otimes y, z)$ is given by $g \mapsto ev_y^z \circ T'(g \otimes id_y)$.
- $(x_0 \oplus x_1, s_i^{x_0,x_1}, p_i^{x_0,x_1})$ is a biproduct of x_0, x_1 (see [3, P.190]).

Obviously these relations are represented by elements of $Mor(D')$. We set

$$\mathcal{F}_{cl}(C) := D'/J.$$

Using the universality of each D_j, one can check the functor : $\mathsf{dgCat}^{\geq 0} \to \mathsf{dgCat}_{cl}^{\geq 0}$ given by $C \mapsto \mathcal{F}_{cl}(C)$ is a left adjoint of the forgetful functor and we have completed the construction of the functor $\mathcal{F}_{cl}$ □

2.3 A model category structure on $\mathsf{dgCat}_{cl}^{\geq 0}$

2.3.1 Limit and colimit

We must show the category $\mathsf{dgCat}_{cl}^{\geq 0}$ is closed under small limits and colimits. Limits are equal to those of underlying dg-categories with the additional structures naturally defined on them. As for colimits, pushouts can be constructed by induction. Let $C_1 \leftarrow C_0 \to C_2$ be a diagram in $\mathsf{dgCat}_{cl}^{\geq 0}$. We put $D_{-1} := C_1 \sqcup_{C_0} C_2$, the pushout in $\mathsf{dgCat}^{\geq 0}$ (see [16]) and attach objects and morphisms step by step similarly to the construction of $\mathcal{F}_{cl}$ using universal dgc's. Infinite coproducts are similar.

2.3.2 Functorial path object

For the path object argument (Theorem 2.1.3), we need a functorial path object in $\mathsf{dgCat}_{cl}^{\geq 0}$, that is, a pair of

- an endo-functor $\mathcal{P} : \mathsf{dgCat}_{cl}^{\geq 0} \to \mathsf{dgCat}_{cl}^{\geq 0}$ and
- a sequence of natural transformations $\{ C \to \mathcal{P}(C) \overset{d_0 \times d_1}{\to} C \times C \}_{C \in \mathsf{dgCat}_{cl}^{\geq 0}}$ which is a factorization of the diagonal, such that $\mathcal{U}(C) \to \mathcal{U}(\mathcal{P}(C)) \to \mathcal{U}(C \times C)$, where $\mathcal{U} : \mathsf{dgCat}_{cl}^{\geq 0} \to \mathsf{dgCat}$ is the forgetfull functor, is a path object diagram in dgCat for any $C \in \mathsf{dgCat}_{cl}^{\geq 0}$.

Note that $k \to \nabla(1,*) \overset{d_0 \times d_1}{\to} k \times k$ is a path object in the category of commutative dg-algebras over k. (For the notations, see subsection 3.1.) We define $\mathcal{P}(C)$ as follows.

- An object of $\mathcal{P}(C)$ is an isomorphism in $Z^0 C$.
- For isomorphisms $f : c_0 \to c_1,\ f' : c_0' \to c_1' \in Z^0 C$

$$Hom_{\mathcal{P}(C)}(f, f') := Hom_C(c_0, c_0') \otimes_k \nabla(1,*)$$

  and the composition is given by $(\beta \otimes \eta) \circ (\alpha \otimes \omega) := (-1)^{\deg \eta \cdot \deg \alpha} (\beta \circ \alpha) \otimes (\eta \cdot \omega)$, where $\alpha \in Hom_C(c_0, c_0')$, $\beta \in Hom_C(c_0', c_0'')$ and $\omega, \eta \in \nabla(1,*)$.

- The additional structures are defined by those of C. For example,

$$\begin{aligned} \mathfrak{Hom}_{\mathcal{P}(C)}(f, f') &:= \mathfrak{Hom}_C(f^{-1}, f') : \mathfrak{Hom}(c_0, c_0') \to \mathfrak{Hom}(c_1, c_1') \\ \mathfrak{Hom}_{\mathcal{P}(C)}(\alpha \otimes \omega, \beta \otimes \eta) &:= (-1)^{\deg \omega \cdot \deg \beta} \mathfrak{Hom}(\alpha, \beta) \otimes (\omega \cdot \eta), \end{aligned}$$

  where $\alpha \in Hom_C(c_0, c_0')$, $\beta \in Hom_C(d_0, d_0')$ and $\omega, \eta \in \nabla(1,*)$.

$d_0 : \mathcal{P}(C) \to C$ is given by $(c_0 \to c_1) \mapsto c_0$ and $id \otimes_k d_0 : Hom(c_0, c_0') \otimes \nabla(1,*) \to Hom(c_0, c_0')$, $d_1 : \mathcal{P}(C) \to C$ is given by $(c_0 \to c_1) \mapsto c_1$ and $(f'_* \circ (f^{-1})^*) \otimes d_1 : Hom(c_0, c_0') \otimes \nabla(1,*) \to Hom(c_1, c_1')$ and $C \to \mathcal{P}(C)$ by $c \mapsto (c \overset{id}{\to} c)$.

Lemma 2.3.1. *With above notations, the sequence*

$$\mathcal{U}C \xrightarrow{\mathcal{U}i} \mathcal{U}\mathcal{P}C \xrightarrow{\mathcal{U}d_0 \times \mathcal{U}d_1} \mathcal{U}C \times \mathcal{U}C$$

is a path object of $\mathcal{U}C$ in dgCat.

Proof. We shall show $\mathcal{U}i$ is a quasi-equivalence and $\mathcal{U}d_0 \times \mathcal{U}d_1$ is a fibration in dgCat. In the following we omit $\mathcal{U}$. Clearly any object $f : c_0 \to c_1 \in \mathcal{P}C$ is isomorphic to id_{c_0} and $k \to \nabla(1,*) \overset{d_0 \times d_1}{\to} k \times k$ is a path object in the category of commutative dg-algebras with the usual model structure so we see that i is a quasi-equivalence and $d_0 \times d_1$ induces surjections on complexes of morphisms. Let $f : c_0 \to c_1 \in \mathcal{P}C$ be an object and $(g, g') : (c_0, c_1) \to (c, c') \in C \times C$ be an isomorphism in $H^0(C \times C) = Z^0(C \times C)$. Note that $(c, c') = (d_0 \times d_1)(g' \circ f \circ g^{-1})$, where $g' \circ f \circ g^{-1}$ is considered as an object of $\mathcal{P}C$. Consider g as an isomorphism $f \to g' \circ f \circ g^{-1} \in Z^0 \mathcal{P}C$ via the injection $Hom_C(c_0, c) \to Hom_C(c_0, c) \otimes_k \nabla(1,*) = Hom_{\mathcal{P}C}(f, g' \circ f \circ g^{-1})$. It is clear that $(d_0 \times d_1) g = (g, g')$ so by above assertion, $d_0 \times d_1$ is a fibration in dgCat. $\square$

Now one can prove the following theorem using the free functor $\mathcal{F}_{cl}^{\geq 0} : \mathsf{dgCat} \to \mathsf{dgCat}_{cl}^{\geq 0}$ and the functorial path object $\mathcal{P}$.

Theorem 2.3.2. *The category $\mathsf{dgCat}_{cl}^{\geq 0}$ has a cofibrantly generated model category structure where weak equivalences and fibrations are defined as follows.*

- *A morphism $F : C \to D \in \mathsf{dgCat}_{cl}^{\geq 0}$ is a weak equivalence if and only if it is a quasi-equivalence.*
- *A morphism $F : C \to D \in \mathsf{dgCat}_{cl}^{\geq 0}$ is a fibration if and only if it satisfies the following two conditions.*
  - *For $c, c' \in Ob(C)$ the morphism $F_{(c,c')} : Hom_C(c, c') \to Hom_D(Fc, Fc')$ is an epimorphism.*
  - *For any $c \in Ob(C)$ and any isomorphism $f : Fc \to d' \in Z^0(D)$, there exists an isomorphism $g : c \to c' \in Z^0(C)$ such that $Z^0(F)(g) = f$.*

Proof. Apply Theorem 2.1.3 to the forgetful functor $\mathcal{U} : \mathsf{dgCat}_{cl}^{\geq 0} \longrightarrow \mathsf{dgCat}$. Note that the fibrations of the statement correspond to those of Theorem 2.1.2 as $H^0C = Z^0C$ for $C \in \mathsf{dgCat}_{cl}^{\geq 0}$. □

We also consider the augmented category. Let $Vect'$ denote the category of all finite dimensional k-vector spaces and k-linear maps. We let $\mathbf{1} = k$ regarded as a k-vector space and fix a 0-dimensional vector space $\mathbf{0}$. With these distinguished objects, $Vect'$ has a closed tensor structure with the the usual operations $\otimes$, $\mathfrak{Hom}$ and $\oplus$. We denote by $Vect$ the full subcategory of $Vect'$ consisting of objects represented by words generated by $\mathbf{1}, \mathbf{0}$ with the operations $\otimes$, $\mathfrak{Hom}$ and $\oplus$. $Vect$ is small and it is (isomorphic to) an initial object of $\mathsf{dgCat}_{cl}^{\geq 0}$.

Definition 2.3.3. *We call the over category $\mathsf{dgCat}_{cl}^{\geq 0}/Vect$ the* category of closed tensor dg-categories with fiber functors *and denote it by $\mathsf{dgCat}_{cl,*}^{\geq 0}$. For an object $C = (C, \omega_C) \in \mathsf{dgCat}_{cl,*}^{\geq 0}$ we call $\omega_C : C \to Vect$ the* fiber functor *of C.*

The following is a corollary of Theorem 2.3.2 (see 1.1.2).

Corollary 2.3.4. *$\mathsf{dgCat}_{cl,*}^{\geq 0}$ has a model category structure induced by that of $\mathsf{dgCat}_{cl}^{\geq 0}$.*

3 The Sullivan-de Rham equivalence for finite fundamental group

The purpose of this section is to prove an equivalence theorem for the homotopy category of spaces whose fundamental group is finite and whose homotopy groups are finite dimensional $\mathbb{Q}$-vector spaces. We call the equivalence the Sullivan-de Rham equivalence because it is a direct generalization of the one in [7].

For an abstract group G, $Rep(G)$ stands for the category of finite dimensional k-linear representations of G whose underlying vector spaces belong to $Vect$. $Rep(G)$ has a closed tensor structure such that the forgetful functor $\omega_G : Rep(G) \to Vect$ is a morphism of closed tensor categories.

3.1 The generalized de Rham functor

In this subsection, we define a Quillen pair

$$T_{dR} : \mathsf{sSet}_* \rightleftarrows (\mathsf{dgCat}_{cl,*}^{\geq 0})^{op} : Sp.$$

We first recall the notion of standard simplicial commutative dga $\nabla(*,*)$ over k from [7, Section 1]. Let $p \geq 0$ and $\nabla(p,*)$ be the commutative graded algebra over k generated by indeterminates $t_0,\dots,t_p$ of degree 0 and $dt_0,\dots,dt_p$ of degree 1 with relations

$$t_0+\cdots+t_p=1, \quad dt_0+\cdots+dt_p=0.$$

We regard $\nabla(p,*)$ as a cdga with the differential given by $d(t_i) := dt_i$. We can define simplicial operators

$$d_i : \nabla(p,*) \to \nabla(p-1,*), \quad s_i : \nabla(p,*) \to \nabla(p+1,*), \quad 0 \leq i \leq p$$

(see [7]) and we also regard $\nabla(*,*)$ as a simplicial commutative dga.

The following definition is adopted in [15]

Definition 3.1.1. *Let $Vect^{iso}$ be the subcategory of $Vect$ consisting of all objects and isomorphisms. Let K be a simplicial set.*

A local system *$\mathscr{L}$ on K is a functor $(\Delta K)^{op} \to Vect^{iso}$ such that for any simplex $\sigma \in \Delta K$, any degeneracy operator s_i and any morphism $f : s_i\sigma \to \sigma$, $\mathscr{L}(\sigma) = \mathscr{L}(s_i(\sigma))$ and $\mathscr{L}(f) = id_{\mathscr{L}(\sigma)}$.*

A morphism of local systems *$\mathscr{L} \to \mathscr{L}'$ is a natural transformation $I \circ \mathscr{L} \Rightarrow I \circ \mathscr{L}' : (\Delta K)^{op} \to Vect$, where $I : Vect^{iso} \to Vect$ is natural inclusion functor. We define the tensor $\mathscr{L} \otimes \mathscr{L}'$, the* internal hom object *$\mathfrak{Hom}(\mathscr{L},\mathscr{L}')$ and the* coproduct *$\mathscr{L} \oplus \mathscr{L}'$ of two local systems $\mathscr{L}$, $\mathscr{L}'$ objectwisely by using those of $Vect$. We denote by $Loc(K)$ be the closed tensor category of local systems on K. If K is pointed, $Loc(K)$ is regarded as a closed tensor category with the fiber functor given by the evaluation at the base point.*

It is well-known that for pointed connected simplicial set K, there exists an equivalence of closed tensor categories $Loc(K) \xrightarrow{\sim} Rep(\pi_1(K))$ which is functorial in K. In the following, we sometimes identify local systems with representations of the fundamental group, fixing such an equivalence.

Definition 3.1.2. *Let K be a simplicial set and $\mathscr{L}$ be a local system on K. The* de Rham complex of *$\mathscr{L}$*-valued polynomial forms *$A_{dR}(K,\mathscr{L}) \in \mathsf{C}(k)$ is defined as follows. For each $q \geq 0$, the degree q part is given by*

$$A^q_{dR}(K,\mathscr{L}) = \lim_{\Delta K^{op}} \nabla(*,q) \otimes_k \mathscr{L}.$$

Here $\nabla(*,q)$ is regarded as a functor from ΔK^{op} to the category of k-vector spaces by composed with the functor $\Delta K^{op} \to \Delta^{op}$, the limit is taken in the category of possibly infinite dimensional k-vector spaces. For $q \leq -1$, we set $A^q_{dR}(K,\mathscr{L}) = 0$. The differential $d : A^q_{dR}(K,\mathscr{L}) \to A^{q+1}_{dR}(K,\mathscr{L})$ is defined from the differential $d : \nabla(*,q) \to \nabla(*,q+1)$.

We shall define the generalized de Rham functor

$$T_{dR} : \mathsf{sSet} \longrightarrow (\mathsf{dgCat}^{\geq 0}_{cl})^{op}.$$

This is a natural generalization of the de Rham functor of [7, Definition 2.1]. For $K \in \mathsf{sSet}$ we define a closed tensor dgc $T_{dR}(K)$ as follows. An object is a local system on K and $Hom_{T_{dR}(K)}(\mathscr{L},\mathscr{L}') = A_{dR}(K, \mathfrak{Hom}(\mathscr{L},\mathscr{L}'))$. The composition is defined from that of the category of vector spaces and the multiplication of $\nabla(*,*)$, i.e.,

$$(\eta \cdot b) \circ (\omega \cdot a) := (\eta \cdot \omega) \cdot (b \circ a)$$

for $\omega, \eta \in \nabla(*,*)$, $a \in \mathfrak{Hom}(\mathscr{L},\mathscr{L}')$ and $b \in \mathfrak{Hom}(\mathscr{L}',\mathscr{L}'')$. The additional structures $\otimes$, $\mathfrak{Hom}$ and $\oplus$ are defined similarly. (We agree that) $T_{dR}(\emptyset)$ is a terminal object of $\mathsf{dgCat}^{\geq 0}_{cl}$ and we always identify

$T_{dR}(*)$ with $Vect$ via the isomorphism $\mathscr{L} \mapsto \mathscr{L}(*)$. For each morphism $f : K \to L \in \mathsf{sSet}$ we associate a morphism of closed tensor dgc's $f^* : T_{dR}(L) \to T_{dR}(K)$ by

$$(\Delta(L)^{op} \overset{\mathscr{L}}{\to} Vect^{iso}) \longmapsto (\Delta(K)^{op} \overset{\Delta f^{op}}{\to} \Delta(L)^{op} \overset{\mathscr{L}}{\to} Vect^{iso}).$$

Thus we have defined a functor $T_{dR} : \mathsf{sSet} \longrightarrow (\mathsf{dgCat}_{cl}^{\geq 0})^{op}$.

Let $C \in \mathsf{dgCat}_{cl}^{\geq 0}$. We define a functor $Sp : (\mathsf{dgCat}_{cl}^{\geq 0})^{op} \to \mathsf{sSet}$ by

$$Sp(C)_n = Hom_{\mathsf{dgCat}_{cl}^{\geq 0}}(C, T_{dR}(\Delta^n))$$

with obvious simplicial operators.

Lemma 3.1.3. *The functor Sp is a right adjoint of T_{dR} and the adjoint pair is a Quillen pair between sSet and $(\mathsf{dgCat}_{cl}^{\geq 0})^{op}$. So it induces a Quillen pair between pointed categories:*

$$T_{dR} : \mathsf{sSet}_* \rightleftarrows (\mathsf{dgCat}_{cl,*}^{\geq 0})^{op} : Sp.$$

Proof. The first assertion is clear. To show the second one, it is enough to examine the generating cofibrations and trivial cofibrations of sSet. One can check easily the condition about lifting of isomorphisms and the proof reduces to the case of constant coefficients, see [7, Section 1, 2]. The third follows from the second. □

3.2 Tannakian dg-categories and equivariant commutative dg-algebras

In this subsection, we introduce the category of Tannakian dg-categories of finite type, which we will prove corresponds to the category $\mathsf{sSet}_*^{f\mathbb{Q}}$ via T_{dR}, and compare it with the category of equivariant dg-algebras. Throughout this subsection, we assume $k = \mathbb{Q}$.

3.2.1 Tannakian dg-categories of finite type

Definition 3.2.1. *Let $(C, \omega_C) \in \mathsf{dgCat}_{cl,*}^{\geq 0}$. We say (C, ω_C) is a* Tannakian dg-category of finite type *if the following conditions are satisfied.*

- *$(Z^0(C), Z^0(\omega))$ is equivalent to $(Rep(G), \omega_G)$, the closed tensor category of finite dimensional representations of G with the forgetful functor. More precisely, there exists a finite chain of morphisms of closed tensor categories with fiber functors which are equivalences of underlying categories:*

$$(Z^0(C), Z^0(\omega)) \overset{\sim}{\to} (T_1, \omega_1) \overset{\sim}{\leftarrow} \cdots \overset{\sim}{\to} (T_n, \omega_n) \overset{\sim}{\leftarrow} (Rep(G), \omega_G).$$

- *For each $c_0, c_1 \in ObC$, $H^1(Hom_C(c_0, c_1)) = 0$ and $H^i(Hom_C(c_0, c_1))$ is finite dimensional for $i \geq 2$*

We denote by Tan^f the full subcategory of $\mathsf{dgCat}_{cl,*}^{\geq 0}$ consisting of Tannakian dgc's of finite type.

Clearly Tan^f is stable under weak equivalences of $\mathsf{dgCat}_{cl,*}^{\geq 0}$.

If we use Tannakian theory, we get an internal characterization of the subcategory Tan^f (which is not used in the rest of the paper, see [10, Theorem 2.11,Proposition 2.20 (a)]):

Proposition 3.2.2. *An object $(C, \omega_C) \in \mathsf{dgCat}_{cl,*}^{\geq 0}$ belongs to Tan^f if and only if the following conditions are satisfied.*

- *The additive category $Z^0(C)$ is an abelian category and the functor $Z^0\omega_C : Z^0(C) \to Vect$ is exact and faithful.*
- $Hom_{Z^0C}(\mathbf{1}, \mathbf{1}) \cong k$.
- *There exists an object $c \in Z^0C$ such that any object of $Z^0(C)$ is a sub-object of a finite coproduct $c^{\oplus N}$ for some N.*
- *For each $c_0, c_1 \in ObC$, $H^1(Hom_C(c_0, c_1)) = 0$ and $H^i(Hom_C(c_0, c_1))$ is finite dimensional for $i \geq 2$.*

3.2.2 Equivariant commutative dg-algebras

Let G be an abstract group. Let $\mathsf{Mod}(G)$ be the category of possibly infinite dimensional right G-modules over k and $\mathsf{dgMod}(G)$ be the category of cochain complexes over $\mathsf{Mod}(G)$. $\mathsf{dgMod}(G)$ has a structure of symmetric monoidal category as usual and we denote by $\mathsf{dgAlg}(G)$ the category of commutative monoids over $\mathsf{dgMod}(G)$. We call an object of $\mathsf{dgAlg}(G)$ a G-equivariant commutative dg-algebra, in short, G-cdga.

$\mathsf{dgMod}(G)$ has a model category structure such that a morphism is a weak equivalence (resp. fibration) if and only if it is a quasi-isomorphism (resp. levelwise epimorphism). The following lemma follows from the path object argument.

Lemma 3.2.3. *$\mathsf{dgAlg}(G)$ has a model structure such that a morphism is a weak equivalence (resp. a fibration) if and only if it is a quasi-isomorphism (resp. levelwise epimorphism).*

Definition 3.2.4. (1) *The* category of equivariant cdga's $\mathsf{EqdgAlg}$ *is defined as follows.*

- *An* object *is a pair (G, A) of a group G and a G-cdga A.*
- *A* morphism *$f : (G, A) \to (H, B)$ is a pair of a group homomorphism $f^{gr} : H \to G$ and a morphism of H-cdga $f : (f^{gr})^*A \to B$.*

We say an equivariant cdga (G, A) is of finite type *if G is a finite group and H^iA is finite dimensional for any i and* 1-connected *if $H^0A \cong k$ and $H^1A \cong 0$. we denote by $\mathsf{EqdgAlg}^f_1$ the full subcategory of $\mathsf{EqdgAlg}$ consisting of 1-connected cdga's of finite type and by $\mathsf{EqdgAlg}^f_{1,*}$ the over category $\mathsf{EqdgAlg}^f_1/(e, k)$, where e is a trivial group.*

(2) *We define the* homotopy category $\mathsf{Ho}(\mathsf{EqdgAlg}^f_{1,*})$ *as the localization of* $\mathsf{EqdgAlg}^f_{1,*}$ *obtained by inverting all the maps whose group homomorphisms are isomorphisms and whose cdga homomorphisms are quasi-isomorphisms.*

(3) *Let $f_1, f_2 : (G, A) \to (H, B)$ be two morphisms of $\mathsf{EqdgAlg}^f_{1,*}$. we say f_1 and f_2 are* right homotopic, *written $\sim_r$ if $f_1^{gr} = f_2^{gr}$ and if f_1, $f_2 : (f_1^{gr})^*A \to B$ are right homotopic as morphisms of $\mathsf{dgAlg}(H)/k$ with respect to its model structure (see [14, DEFINITION 1.2.4]).*

The following lemma is proved by arguments similar to the proofs of [14, Proposition 1.2.5, Theorem 1.2.10].

Lemma 3.2.5. *Let $(G, A), (H, B) \in \mathsf{EqdgAlg}^f_{1,*}$ and suppose A is cofibrant as an object of $\mathsf{dgAlg}(G)$. Then the relation $\sim_r$ on $Hom_{\mathsf{EqdgAlg}^f_{1,*}}((G, A), (H, B))$ is an equivalence relation and there exists a bijection*

$$Hom_{\mathsf{EqdgAlg}^f_{1,*}}((G, A), (H, B))/\sim_r \cong Hom_{\mathsf{Ho}(\mathsf{EqdgAlg}^f_{1,*})}((G, A), (H, B))$$

which takes a class represented by a map $f : (G, A) \to (H, B)$ to the image of f by the canonical functor $\mathsf{EqdgAlg}^f_{1,*} \to \mathsf{Ho}(\mathsf{EqdgAlg}^f_{1,*})$.

3.2.3 Comparison

We shall define two functors

$$\mathsf{T}, \mathsf{T}^c : \mathsf{EqdgAlg}^f_{1,*} \longrightarrow \mathsf{dgCat}^{\geq 0}_{cl,*}.$$

Let $A = (G, A) \in \mathsf{EqdgAlg}^f_{1,*}$. For $V \in Ob(Rep(G))$ we define a complex $A \otimes^G V$ by

$$A \otimes^G V := \{\, \Sigma_j a_j \otimes v_j \in A \otimes_k V \mid \Sigma_j (a_j \cdot g) \otimes v_j = \Sigma_j a_j \otimes (g \cdot v_j) \text{ for } \forall g \in G. \,\}$$

and set

$$Ob(\mathsf{T}A) := Ob(Rep(G)), \quad Hom_{\mathsf{T}A}(V, W) := A \otimes^G \mathfrak{Hom}_{Rep(G)}(V, W),$$

where $\mathfrak{Hom}_{Rep(G)}$ is the internal hom of $Rep(G)$. We define composition, closed tensor structure of $\mathsf{T}(A)$ using corresponding structures of $Rep(G)$ and the multiplication of A. A morphism $f : (G, A) \to (H, B)$ of $\mathsf{EqdgAlg}^f_{1,*}$ gives a functor $(f^{gr})^* : Rep(G) \to Rep(H)$ so f induces a morphism $\mathsf{T}f : \mathsf{T}A \to \mathsf{T}B$ of closed tensor dg-categories. The augmentation $A \to k$ defines a fiber functor $\mathsf{T}A \to \mathsf{T}k \cong Vect$ and we have defined a functor $\mathsf{T} : \mathsf{EqdgAlg}^f_{1,*} \to \mathsf{dgCat}^{\geq 0}_{cl,*}$.

Example 3.2.6. *Let $(G, k) \in \mathsf{EqdgAlg}^f_{1,*}$ denote the equivariant cdga whose underlying cdga is k and whose group action is trivial. $\mathsf{T}(G, k)$ is isomorphic to $(Rep(G), \omega_G)$.*

T^c is defined as follows. Let $W_{cl}(Ob(Rep(G)))$ be the set of words freely generated by $Ob(Rep(G)) \sqcup \{\mathbf{1}, \mathbf{0}\}$ with operations $\widehat{\otimes}$, $\widehat{\oplus}$ and $\widehat{\mathfrak{Hom}}$ (see sub-subsection 2.2.2). Let $R : W_{cl}(Ob(Rep(G))) \to Ob(Rep(G))$ be the function defined inductively, by

- $RX = X$ for $X \in Ob(Rep(G))$,
- $R(X \widehat{\otimes} Y) = (RX) \otimes (RY)$, $R(\widehat{\mathfrak{Hom}}(X, Y)) = \mathfrak{Hom}(RX, RY)$, and $R(X \widehat{\oplus} Y) = (RX) \oplus (RY)$

We define T^c as the "pullback" of T by R. Precisely, we set

$$Ob\mathsf{T}^c A := W_{cl}(Ob(Rep(G))), \quad Hom_{\mathsf{T}^c A}(X, Y) := Hom_{\mathsf{T}A}(RX, RY)$$

Also the closed tensor structure on $\mathsf{T}^c A$ is defined by "pullback" by R. For example, the tensor structure is given by the operation $\widehat{\otimes}$ of $W_{cl}(Ob(Rep(G)))$. R defines a morphism $R_A : \mathsf{T}^c A \to \mathsf{T}A$ and we define the fiber functor of $\mathsf{T}^c A$ by the composition $\mathsf{T}^c A \overset{R_A}{\to} \mathsf{T}A \overset{\omega_{\mathsf{T}A}}{\to} Vect$. Thus, we have defined a functor $\mathsf{T}^c : \mathsf{EqdgAlg}^f_{1,*} \to \mathsf{dgCat}^{\geq 0}_{cl,*}$.

Obviously, $\mathsf{T}^c A$ is naturally equivalent to $\mathsf{T}A$ via R_A and $\mathsf{T}A$ is simpler but $\mathsf{T}^c A$ is convenient to model categorical arguments because of the following:

Lemma 3.2.7. *For a finite group G, $\mathsf{T}^c(G, k)$ is cofibrant in $\mathsf{dgCat}^{\geq 0}_{cl,*}$.*

Proof. For closed tensor categories C, D, consider the lifting problem

$$\begin{tikzcd} & C \arrow[d, "P"] \\ \mathsf{T}^c(G,k) \arrow[ur, dashed] \arrow[r, "F"] & D, \end{tikzcd}$$

where F and P are morphisms of closed tensor categories and P is an equivalence of categories which induces a surjective map on the sets of objects. We can find a lifting $\widetilde{F} : \mathsf{T}^c(G, k) \to C$ as a functor but $\widetilde{F}$ may not be a morphism of closed tensor categories. For example, $\widetilde{F}(X)\otimes\widetilde{F}(Y)$ and $\widetilde{F}(X\otimes Y)$ are isomorphic, but not always equal. We shall modify $\widetilde{F}$ so that it becomes a morphism of closed tensor categories. As $Ob(\mathsf{T}^c(G,k))$ is freely generated by $Ob(Rep(G))\sqcup\{\mathbf{1},\mathbf{0}\}$, we can define a function $\widetilde{F}':Ob(\mathsf{T}^c(G,k))\to Ob(C)$ such that $\widetilde{F}'(X)=\widetilde{F}(X)$ for $X\in Ob(Rep(G))$ and $\widetilde{F}'$ preserves $\otimes,\mathfrak{Hom},\oplus,\mathbf{1}$ and $\mathbf{0}$. For a morphism $f\in Hom_{\mathsf{T}^c(G,k)}(X,Y)$, we define $\widetilde{F}'(f)\in Hom_C(\widetilde{F}'X,\widetilde{F}'Y)$ as the composition

$$\widetilde{F}'X\overset{\varphi_X}{\to}\widetilde{F}X\overset{\widetilde{F}(f)}{\to}\widetilde{F}Y\overset{\varphi_Y^{-1}}{\to}\widetilde{F}'Y.$$

Here, $\varphi_Z:\widetilde{F}'(Z)\to\widetilde{F}(Z)$ is the unique isomorphism such that $P(\varphi_Z)=id_{F(Z)}$. Thus we have defined a functor $\widetilde{F}':\mathsf{T}^c(G,k)\to C$. This is clearly a morphism of closed tensor categories and we have completed the proof. $\square$

To see some basic properties of T, we need a few elementary preparations. For a finite group G let V_r^G be the vector spaces of k-valued functions on G. (This vector space does not belong to *Vect* but it is isomorphic to an object of *Vect* and so we fix an isomorphism and we deal with V_r^G as if it belonged to *Vect* via the isomorphism.) There are two actions ρ, ϱ of G on V_r^G.

$$[\rho(g)\alpha](g')=\alpha(g'g),\quad [\varrho(g)\alpha](g')=\alpha(gg'),\qquad \alpha:G\to k\in V_r^G.$$

ρ is a left action and ϱ is a right action. We call the representation (V_r^G,ρ) the regular representation of G and sometimes we omit ρ. Let $V\in Ob(Rep(G))$. In the following we use the homomorphism

$$\phi_V:V\to\mathfrak{Hom}_{Rep(G)}(V_u^\vee,V_r^G),\quad \phi_V(v)(v')(g)=\langle v',gv\rangle$$

where V_u is the trivial representation with the same underlying vector space as V. This is a monomorphism and has a retraction.

Lemma 3.2.8. (1) *For each* $(G,A)\in\mathsf{EqdgAlg}^f_{1,*}$*, the morphism of complexes*

$$A\longrightarrow A\otimes^G V_r^G,\qquad a\longmapsto\Sigma_{g\in G}a\cdot g\otimes\delta_g,$$

where δ_g *is the* δ*-function at* $g\in G$*, is an isomorphism.*
(2) *Let* $f:(G,A)\to(G,B)$ *be a morphism of* $\mathsf{EqdgAlg}^f_{1,*}$ *such that* $f^{gr}=id$*. If* f *is a quasi-isomorphism of* G*-cdga's,* $\mathsf{T}f:\mathsf{T}A\to\mathsf{T}B$ *is a quasi-equivalence.*
(3) $\mathsf{T}A$ *is a Tannakian dgc of finite type for any* $A\in\mathsf{EqdgAlg}^f_{1,*}$*.*

Proof. (1) is easy. (2) is a consequence of (1) and the fact that for any $V\in Rep(G)$, $A\otimes^G V$ is a retract of $A\otimes^G\mathfrak{Hom}_{Rep(G)}(V_u^\vee,V_r^G)\cong A^{\oplus\dim V}$. (3) follows from the fact that (G,A) is quasi-isomorphic to a G-cdga (G,M) such that $M^0\cong k$, $M^1\cong 0$. $\square$

Let (C,ω_C), $(D,\omega_D)\in\mathsf{dgCat}^{\geq0}_{cl,*}$ and F, $F':(C,\omega_C)\to(D,\omega_D)$ be two morphisms. We say F and F' are 2-isomorphic if there exists a natural isomorphism $i_-:F\Rightarrow F'$ such that $i_{c_0\otimes c_1}=i_{c_0}\otimes i_{c_1}$ and $\omega_D(i_c)=id_{\omega_C(c)}$. If C and D are Tannakian dgc's of finite type, i is unique if it exists.

(1) of the following is a rewrite of [10, Proposition 2.8] for the case of finite group and (2) follows from (1).

Theorem 3.2.9 ([10])**.** *Let* G *be a finite group. Let* $\omega=\omega_{\mathsf{T}^c(G,k)}$*. Let* $Aut^{\otimes}(\omega)$ *be the group of tensor preserving automorphisms of* ω *i.e.,*

$$Aut^{\otimes}(\omega)=\{\alpha:\omega\Rightarrow\omega\,|\,\alpha_{X\otimes Y}=\alpha_X\otimes\alpha_Y\}.$$

(1) *The homomorphism* $\phi_G : G \to Aut^{\otimes}(\omega)$ *defined by* $\phi_G(g)_X = r_{RX}(g) : \omega(X) \to \omega(X)$*, where* r_{RX} *is the action of* G *endowed with the representation* RX*, is an isomorphism of groups*
(2) *Let* H *be another finite group. Set* $\omega' = \omega_{\mathsf{T}^c(H,k)}$*. Let* $F : \mathsf{T}^c(G,k) \to \mathsf{T}^c(H,k)$ *be a morphism of closed tensor category such that* $\omega' \circ F = \omega$*.* F *is 2-isomorphic to* $\mathsf{T}^c(f)$ *for some* $f : (G,k) \to (H,k) \in \mathsf{EqdgAlg}^f_{1,*}$*.*

The following theorem says the category of Tannakian dgc's of finite type and the category of 1-connected augmented equivariant dg-algebras of finite type are essentially the same. In the proofs of this theorem and Lemma 3.2.11 we need internal hom functors $\mathfrak{Hom}$.

Theorem 3.2.10. (1) *The functor* $\mathsf{T}^c : \mathsf{EqdgAlg}^f_{1,*} \to \mathsf{Tan}^f$ *is fully faithful up to 2-isomorphisms. More precisely, for any* A*,* $B \in \mathsf{EqdgAlg}^f_{1,*}$ *and* $F : \mathsf{T}^c A \to \mathsf{T}^c B \in \mathsf{Tan}^f$ *there exists a unique morphism* $f : A \to B \in \mathsf{EqdgAlg}^f_{1,*}$ *such that* $\mathsf{T}^c f$ *is 2-isomorphic to* F*.*
(2) *Any object of* Tan^f *is equivalent to* $\mathsf{T}A$ *(and* $\mathsf{T}^c A$*) for some* $A \in \mathsf{EqdgAlg}^f_{1,*}$*.*
(3) *Let* $f : (G,A) \to (G,B) \in \mathsf{EqdgAlg}^f_{1,*}$ *be a morphism such that* f^{gr} *is the identity. Suppose* f *is a cofibration as a morphism of* $\mathsf{dgAlg}(G)$*. Then* $\mathsf{T}^c f : \mathsf{T}^c A \to \mathsf{T}^c B$ *is a cofibration in* $\mathsf{dgCat}^{\geq 0}_{cl,*}$*. In particular, if* $(G,A) \in \mathsf{EqdgAlg}^f_{1,*}$ *is cofibrant as an object of* $\mathsf{dgAlg}(G)$*,* $\mathsf{T}^c(G,A)$ *is cofibrant in* $\mathsf{dgCat}^{\geq 0}_{cl,*}$*.*
(4) T *and* T^c *induces an equivalence of categories* $\mathsf{Ho}(\mathsf{EqdgAlg}^f_{1,*}) \simeq \mathsf{Ho}(\mathsf{Tan}^f)$*.*

Proof. (1) Let $A = (G,A)$ and $B = (H,B)$. We show surjectivity up to 2-isomorphisms. By Theorem 3.2.9, F is 2-isomorphic to a morphism F' such that $Z^0(F') : \mathsf{T}^c(G,k) \to \mathsf{T}^c(H,k)$ is $\mathsf{T}^c(f^{gr})$ for some group homomorphism $f^{gr} : H \to G$. So we may replace F by such F'.

If $Z^0(F)$ is fixed as above, F is determined by

$$F_{(\mathbf{1},V_r^G)} : Hom_{\mathsf{T}^c A}(\mathbf{1}, V_r^G) \to Hom_{\mathsf{T}^c B}(\mathbf{1}, (f^{gr})^* V_r^G)$$

Let $A' := Hom_{\mathsf{T}A}(\mathbf{1}, V_r^G)$. The right action ϱ on V_r^G defines a right action of G on A and pointwise multiplication $V_r^G \otimes V_r^G \to V_r^G$ defines a structure of cdga on A'. Thus, the tensor structure on $\mathsf{T}^c(A)$ defines a G-cdga structure on A' and the isomorphism of Lemma 3.2.8 $A \to A'$ is an isomorphism of G-cdga's. Composing $F_{(\mathbf{1},V_r^G)}$ with the homomorphism

$$Hom_{\mathsf{T}^c B}(\mathbf{1}, (f^{gr})^* V_r^G) \to Hom_{\mathsf{T}^c B}(\mathbf{1}, V_r^H) =: B'$$

induced by $(f^{gr})^* : (f^{gr})^* V_r^G \to V_r^H \in Rep(H)$, we get a morphism of augmented equivariant cdga's $f' : A' \to B'$. Composing with the isomorphisms $A \cong A'$, $B \cong B'$ defined in Lemma 3.2.8, we get $f : A \to B$ such that $\mathsf{T}^c f = F$. The injectivity is clear from the above argument.

(2) Let $T \in \mathsf{Tan}^f$. We may assume $Z^0 T = Rep(G)$ where G is a finite group. Let $V \in Rep(G)$. There is a G-bimodule structure on $\mathfrak{Hom}(V_u^\vee, V_r^G)$ determined by the right action on V_r^G and the left action on V_u. The image of ϕ_V is $\{\alpha \in \mathfrak{Hom}(V_u^\vee, V_r^G) | g \cdot \alpha = \alpha \cdot g \text{ for } \forall g\}$. So if we put $A := Hom_T(\mathbf{1}, V_r^G)$, the homomorphism $A \otimes_k V_u \cong Hom_T(\mathbf{1}, V_u \otimes V_r^G) \cong Hom_T(\mathbf{1}, \mathfrak{Hom}(V_u^\vee, V_r^G))$ induces an isomorphism of complexes $A \otimes^G V \cong Hom_T(\mathbf{1}, V)$ via ϕ_V. The composition $A \otimes^G \mathfrak{Hom}(V,W) \cong Hom_T(\mathbf{1}, \mathfrak{Hom}(V,W)) \cong Hom_T(V,W)$ defines a morphism $\mathsf{T}A \to T$ of Tannakian dgc's. Various naturalities ensure this is well-defined and this is clearly an equivalence.

(3) Let $\mathsf{T}A \rightarrowtail T \overset{\sim}{\twoheadrightarrow} \mathsf{T}B$ be a factorization in $\mathsf{dgCat}^{\geq 0}_{cl,*}$. Put $B' := Hom_T(\mathbf{1}, V_r^G)$. We regard B' as a G-cdga which has an augmentation $B' \to k$. By an argument similar to the proof of (2), one can take morphisms $i : A \to B'$, $p : B' \to B \in \mathsf{dgAlg}(G)/k$ and $F : \mathsf{T}B' \to T \in \mathsf{Tan}^f$ such that the composition $\mathsf{T}A \xrightarrow{\mathsf{T}i} \mathsf{T}B' \to T'$ is equal to the morphism of the factorization$\mathsf{T}A \to T'$ and the composition

$\mathsf{T}B' \to T' \to \mathsf{T}B$ is equal to $\mathsf{T}B' \xrightarrow{\mathsf{T}p} \mathsf{T}B$. Then one can easily see F has a retraction which preserves the morphisms from $\mathsf{T}A$. So $\mathsf{T}i : \mathsf{T}A \to \mathsf{T}B'$ is a cofibration in $\mathsf{dgCat}^{\geq 0}_{cl,*}$. One can see $\mathsf{T}f : \mathsf{T}A \to \mathsf{T}B$ is a retract of $\mathsf{T}i$ by using a lift of the following diagram.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B' \arrow[d] \\ B \arrow[r, "id"] \arrow[ur, dashed] & B \end{tikzcd}$$

So it is a cofibration in $\mathsf{dgCat}^{\geq 0}_{cl,*}$. The latter claim follows from Lemma 3.2.7.

(4) We show the statement about T^c. Then the one about T follows from it. By (2) it is enough to show the map

$$\mathsf{T}^c_{(A,B)} : Hom_{\mathsf{Ho}(\mathsf{EqdgAlg}^f_{1,*})}(A, B) \longrightarrow Hom_{\mathsf{Ho}(\mathsf{dgCat}^{\geq 0}_{cl,*})}(\mathsf{T}^c A, \mathsf{T}^c B)$$

is a bijection for each $A, B \in \mathsf{EqdgAlg}^f_{1,*}$. We may assume A is cofibrant as G-cdga and $\mathsf{T}^c B$ is fibrant. Then, by (3), Lemma 3.2.5 and [14, Theorem 1.2.10,(ii)],the above sets of morphisms in homotopy categories are identified with the sets of right homotopy classes of morphisms. It is easy to see that for two morphisms $f_1, f_2 : (G, A) \to (H, B) \in \mathsf{EqdgAlg}^f_{1,*}$ f_1 and f_2 are right homotopic if and only if $\mathsf{T}f_1$ and $\mathsf{T}f_2$ are right homotopic in $\mathsf{dgCat}^{\geq 0}_{cl,*}$ and 2-isomorphic morphisms in $\mathsf{dgCat}^{\geq 0}_{cl,*}$ are right homotopic (see sub-subsection 2.3.2). Then the claim follows from (1). □

The following lemma is used in next subsection.

Lemma 3.2.11. *Let $(G, A) \in \mathsf{EqdgAlg}^f_{1,*}$. Let $\tilde{T}$ be an object of Tan^f defined by $Ob(\tilde{T}) = Ob(Vect)$ and $Hom_{\tilde{T}}(V, W) = A \otimes_k Hom_{Vect}(V, W)$. Then there is a commutative diagram in Tan^f*

$$\begin{tikzcd} \mathsf{T}^c(G, k) \arrow[r] \arrow[d] & Vect \arrow[d] \\ \mathsf{T}^c(G, A) \arrow[r] & \tilde{T} \end{tikzcd}$$

where the left vertical morphism is induced by the unit $k \to A$ and the bottom horizontal morphism is given by inclusion $A \otimes^G \mathfrak{Hom}_{Rep(G)}(RX, RY) \subset A \otimes Hom_{Vect}(\omega(X), \omega(Y))$ for each $X, Y \in Ob(\mathsf{T}^c(G, A))$. This diagram is a pushout diagram in $\mathsf{dgCat}^{\geq 0}_{cl,*}$ and a homotopy pushout diagram.

Proof. We show the first assertion. The second one follows from it, Lemma 3.2.7, and Theorem 3.2.10, (3) (see also [14, Lemma 5.2.6]). Let

$$\begin{tikzcd} \mathsf{T}^c(G, k) \arrow[r] \arrow[d] & Vect \arrow[d] \\ \mathsf{T}^c(G, A) \arrow[r, "F"] & C \end{tikzcd}$$

be a commutative diagram in $\mathsf{dgCat}^{\geq 0}_{cl,*}$. We define a homomorphism of complexes

$$\tilde{F}_{(\mathbf{1}, \omega(V_r^G) \otimes V)} : Hom_{\tilde{T}}(\mathbf{1}, \omega(V_r^G) \otimes V) \to Hom_C(\mathbf{1}, \omega(V_r^G) \otimes V)$$

for $V \in Ob(Vect)$. Let $(\Sigma_{g \in G} a_g \otimes \delta_g) \otimes v \in Hom_{\tilde{T}}(\mathbf{1}, \omega(V_r^G) \otimes V) \cong (A \otimes \omega(V_r^G)) \otimes V$ where $a_g \in A$ and $v \in V$. Then $(\Sigma_{g' \in G} a_g \cdot g' \otimes \delta_{g'}) \otimes v$ is regarded as an element of $Hom_{\mathsf{T}A}(\mathbf{1}, V_r^G \otimes V)$. We set

$$\tilde{F}((\Sigma_g a_g \otimes \delta_g) \otimes v) := \Sigma_g f_g \circ F((\Sigma_{g'} a_g \cdot g' \otimes \delta_{g'}) \otimes v)$$

where $f_g : \omega(V_r^G) \to \omega(V_r^G) \in \mathit{Vect}$ is

$$f_g(\delta_h) = \begin{cases} \delta_g & \text{if } h = e \\ 0 & \text{otherwise} \end{cases}$$

In general, one defines $\tilde{F}_{(V,W)} : \mathit{Hom}_{\tilde{T}}(V,W) \to \mathit{Hom}_C(V,W)$ using the embedding $\mathfrak{Hom}(V,W) \to \mathfrak{Hom}((\mathfrak{Hom}(V,W))^\vee, \omega(V_r^G)) \cong \mathfrak{Hom}(V,W) \otimes \omega(V_r^G)$. One can easily check $\tilde{F}_{(V,W)}$'s defines functor $\tilde{F} : \tilde{T} \to C \in \mathsf{dgCat}^{\geq 0}_{cl,*}$ and $\tilde{F}$ makes appropriate diagrams commutative. $\square$

Let $L \in \mathsf{sSet}_*^{f\mathbb{Q}}$. We take the universal covering $\widetilde{L} \to L$. The polynomial de Rham algebra $A_{dR}(\widetilde{L})$ has a natural action of $\pi_1(L)$ induced by the action on $\widetilde{L}$. The construction $L \mapsto (\pi_1(L), A_{dR}(\widetilde{L}))$ defines a functor $\widetilde{A_{dR}} : \mathsf{sSet}_*^{f\mathbb{Q}} \to (\mathsf{EqdgAlg}^f_{1,*})^{op}$. (Here, $\widetilde{L}$ is taken functorially in L.)

We shall compare two functors T_{dR} and $\widetilde{A_{dR}}$.

Proposition 3.2.12. *Let S be either sSet_*^f or $\mathsf{sSet}_*^{f\mathbb{Q}}$ (see 1.1.2). Consider the following diagram.*

$$\begin{tikzcd} & (\mathsf{EqdgAlg}^f_{1,*})^{op} \arrow[d, "\mathsf{T}"] \\ \mathsf{S} \arrow[ur, "\widetilde{A_{dR}}"] \arrow[r, "T_{dR}"'] & (\mathsf{Tan}^f)^{op} \end{tikzcd}$$

There exists a natural transformation $\Phi : T_{dR} \Rightarrow \mathsf{T} \circ \widetilde{A_{dR}}$ such that for each $L \in \mathsf{sSet}_*^{f\mathbb{Q}}$, $\Phi_L : T_{dR}(L) \to \mathsf{T} \circ \widetilde{A_{dR}}(L)$ is an equivalence of underlying categories.

Proof. The projection $p : \widetilde{L} \to L$ defines a morphism $p^* : T_{dR}(L) \to T_{dR}(\widetilde{L}) \simeq \mathit{Vect} \otimes A_{dR}(\widetilde{L})$, where the closed tensor dg-category $\mathit{Vect} \otimes A_{dR}(\widetilde{L})$ is given by $\mathit{Ob}(\mathit{Vect} \otimes A_{dR}(\widetilde{L})) = \mathit{Ob}(\mathit{Vect})$ and $\mathit{Hom}_{\mathit{Vect} \otimes A_{dR}(\widetilde{L})}(V,W) = \mathit{Hom}_{\mathit{Vect}}(V,W) \otimes_k A_{dR}(\widetilde{L})$. For two representations $V, W \in \mathit{Rep}(\pi_1(L))$, the morphism

$$p^*_{(V,W)} : \mathit{Hom}_{T_{dR}(L)}(V,W) \to \mathit{Hom}_{\mathit{Vect}}(V,W) \otimes A_{dR}(\widetilde{L})$$

is a monomorphism and its image is precisely $\mathfrak{Hom}(V,W) \otimes^{\pi_1(L)} \widetilde{A_{dR}}(L)$ so p^* defines the required natural transformation. $\square$

In view of this proposition, we can produce some examples.

Example 3.2.13. *Let G be a finite group and L be a $K(G,1)$-space. $\widetilde{A_{dR}}(L)$ is quasi-isomorphic to (G,k) so $T_{dR}(L)$ is quasi-equivalent to $\mathsf{T}(G,k) \cong \mathit{Rep}(G)$.*

Example 3.2.14. *Let L be the 2-dimensional real projective space $\mathbb{R}P^2$. $\widetilde{A_{dR}}(L)$ is quasi-isomorphic to $(\mathbb{Z}/2, M)$ where M is a cdga freely generated by two generators t, s with $\deg t = 2$, $\deg s = 3$ as a commutative graded algebra and the differential is given by $d(t) = 0$, $d(s) = t^2$. $\mathbb{Z}/2$ acts on M by $g \cdot t = -t$ and $g \cdot s = s$ (g is the generator). $T_{dR}(L)$ is quasi-equivalent to $\mathsf{T}(\mathbb{Z}/2, M)$. By definition, $Z^0(\mathsf{T}(\mathbb{Z}/2, M)) \cong \mathit{Rep}(\mathbb{Z}/2)$. We shall describe the complex of morphisms $\mathit{Hom}_{\mathsf{T}(\mathbb{Z}/2,M)}(\mathbf{1}, V)$ for each irreducible representation V. Let V_- be the 1-dimensional representation where g acts by the multiplication of -1.*

	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$\mathit{Hom}_{\mathsf{T}(\mathbb{Z}/2,M)}(\mathbf{1},\mathbf{1}) =$	$\mathbb{Q}$	0	0	$\mathbb{Q}s$	$\mathbb{Q}t^2$	0	0	$\mathbb{Q}st^2$
$\mathit{Hom}_{\mathsf{T}(\mathbb{Z}/2,M)}(\mathbf{1},V_-) =$	0	0	$\mathbb{Q}t$	0	0	$\mathbb{Q}st$	$\mathbb{Q}t^3$	0
$M =$	$\mathbb{Q}$	0	$\mathbb{Q}t$	$\mathbb{Q}s$	$\mathbb{Q}t^2$	$\mathbb{Q}st$	$\mathbb{Q}t^3$	$\mathbb{Q}st^2$

3.3 The Sullivan-de Rham equivalence theorem for finite fundamental group

In this subsection, we prove Theorem 1.0.2.

We first modify the right adjoint $Sp : (\mathsf{dgCat}^{\geq 0}_{cl,*})^{op} \longrightarrow \mathsf{sSet}_*$. Let sSet^c_* be the full subcategory of sSet_* consisting of connected pointed simplicial sets. The author cannot prove the image of $\mathsf{Ho}(\mathsf{Tan}^f)$ by the functor $\mathbb{R}Sp : \mathsf{Ho}(\mathsf{dgCat}^{\geq 0}_{cl,*})^{op} \longrightarrow \mathsf{Ho}(\mathsf{sSet}_*)$ is contained in $\mathsf{Ho}(\mathsf{sSet}^c_*)$. We define a functor $Sp_0 : (\mathsf{dgCat}^{\geq 0}_{cl,*})^{op} \to \mathsf{sSet}^c_*$ by saying that $Sp_0(C)$ is the connected component of $Sp(C)$ containing the base point for each $C \in \mathsf{dgCat}^{\geq 0}_{cl,*}$. There is an obvious adjunction

$$T_{dR} : \mathsf{sSet}^c_* \rightleftarrows (\mathsf{dgCat}^{\geq 0}_{cl,*})^{op} : Sp_0.$$

This gives derived adjunction $T_{dR} : \mathsf{Ho}(\mathsf{sSet}^c_*) \rightleftarrows \mathsf{Ho}(\mathsf{dgCat}^{\geq 0}_{cl,*})^{op} : \mathbb{R}Sp_0$.

Theorem 3.3.1. *Suppose $k = \mathbb{Q}$.*
(1) *The left Quillen functor $T_{dR} : \mathsf{sSet}_* \to (\mathsf{dgCat}^{\geq 0}_{cl,*})^{op}$ induces an equivalence between homotopy categories:*

$$\mathsf{Ho}(\mathsf{sSet}^{f\mathbb{Q}}_*) \xrightarrow{\sim} \mathsf{Ho}(\mathsf{Tan}^f)^{op} .$$

(2) *Let $K \in \mathsf{sSet}^f_*$. The adjunction map*

$$K \longrightarrow \mathbb{R}Sp_0 T_{dR}(K).$$

is a fiberwise rationalization.

To show this theorem, we need the following lemma and corollary.

Lemma 3.3.2. *Let G be a finite group and K be a $K(G,1)$-space. The unit of the adjunction $K \to \mathbb{R}Sp_0 T_{dR}(K)$ is a weak equivalence of simplicial sets.*

Proof. $\tilde{K}$ is contractible so $T_{dR}(K)$ is quasi-equivalent to $\mathsf{T}^c(G,k)$. As $\mathsf{T}^c(G,k)$ is cofibrant, the morphism $K \to \mathbb{R}Sp_0 T_{dR}(K)$ is weak equivalent to $K \to Sp_0\mathsf{T}(G,k)$ which is the adjoint of the composition

$$\mathsf{T}^c(G,k) \xrightarrow{\sim} Rep(G) \xrightarrow{\sim} Loc(K) \cong Z^0 T_{dR}(K) \to T_{dR}(K).$$

One can see $\pi_i(\mathbb{R}Sp_0 T_{dR}(K)) = 0$ for $i \geq 2$ by the adjunction. So it is enough to show $K \to Sp_0\mathsf{T}^c(G,k)$ gives an isomorphism of π_1. We may assume $K = N(G)$, the nerve of G. Both K and $Sp_0\mathsf{T}^c(G,k)$ are fibrant, one can check this explicitly. A representative of a class in $\pi_1(Sp_0\mathsf{T}({}^cG,k))$ is a morphism $F : \mathsf{T}^c(G,k) \to T_{dR}(\Delta^1)$ of $\mathsf{dgCat}^{\geq 0}_{cl}$ such that $d_0 \circ F = d_1 \circ F = \omega_{\mathsf{T}^c(G,k)} : \mathsf{T}^c(G,k) \to Vect$. We define an element $\alpha_F \in Aut^{\otimes}(\omega_{\mathsf{T}^c(G,k)})$ by $\alpha_F(V) :=$ the composition $F(V)_1 \overset{(d_0)^{-1}}{\to} F(V)_\sigma \overset{d_1}{\to} F(V)_0$ for $V \in Ob(\mathsf{T}^c(G,k))$ (σ is the non-degenerate 1-simplex of Δ^1). α_F corresponds to some $g \in G$ via canonical isomorphism $G \cong Aut^{\otimes}(\omega_{\mathsf{T}^c(G,k)})$ (see Theorem 3.2.9). One can see F represents the same class as $Ev_g : \mathsf{T}^c(G,k) \to T_{dR}(\Delta^1)$, the evaluation at the edge corresponding to g and the assertion follows. □

The following collorary follows from Proposition 3.2.12 and Lemma 3.2.11.

Corollary 3.3.3. *Let $L \in \mathsf{sSet}^{f\mathbb{Q}}_*$. Consider a homotopy fiber sequence*

$$\widetilde{L} \longrightarrow L \longrightarrow K(\pi_1(L),1)$$

where the right map induces isomorophism of π_1*. The corresponding sequence*

$$T_{dR}(K(\pi_1(L),1)) \longrightarrow T_{dR}(L) \longrightarrow T_{dR}(\widetilde{L})$$

is a homotopy cofiber sequence in $\mathsf{dgCat}^{\geq 0}_{cl,*}$.

□

Now we shall prove Theorem 3.3.1, (1). Let $L \in \mathsf{sSet}^{f\mathbb{Q}}_*$ and K be a $K(\pi_1(L),1)$-space. Let $\widetilde{L} \to L \to K$ be a homotopy fiber sequence where the map $L \to K$ induces an isomorphism of π_1. Consider the following diagram.

$$\begin{array}{ccccc}
\widetilde{L} & \longrightarrow & L & \longrightarrow & K \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}Sp_0T_{dR}(\widetilde{L}) & \longrightarrow & \mathbb{R}Sp_0T_{dR}(L) & \longrightarrow & \mathbb{R}Sp_0T_{dR}(K)
\end{array}$$

The left vertical arrow is a weak equivalence by original Sullivan's theory and the right vertical one is a weak equivalence by Lemma 3.3.2. The bottom horizontal sequence is a homotopy fiber sequence by Corollary 3.3.3 and so is the top horizontal one by definition. Hence the middle vertical arrow is a weak equivalence. Thus $T_{dR} : \mathsf{Ho}(\mathsf{sSet}^{f\mathbb{Q}}_*) \to \mathsf{Ho}(\mathsf{Tan}^f)^{op}$ is fully faithful. Essential surjectivity follows from a similar argument and Theorem 3.2.10. (2) follows from the above proof. □

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